Proof. Statement (4.2.7) is an immediate consequence of (4.2.5). To prove (4.2.8), suppose that the basic columns in $\mathbf{A}$ are in positions $b_1, b_2, \ldots, b_r$, and the nonbasic columns occupy positions $n_1, n_2, \ldots, n_t$, and let $\mathbf{Q}_1$ be the permutation matrix that permutes all of the basic columns in $\mathbf{A}$ to the left-hand side so that $\mathbf{AQ}_1 = (\mathbf{B}_{m\times r} \quad \mathbf{N}_{m\times t})$, where $\mathbf{B}$ contains the basic columns and $\mathbf{N}$ contains the nonbasic columns. Since the nonbasic columns are linear combinations of the basic columns—recall (2.2.3)—we can annihilate the nonbasic columns in $\mathbf{N}$ using elementary column operations. In other words, there is a nonsingular matrix $\mathbf{Q}_2$ such that $(\mathbf{B} \quad \mathbf{N})\mathbf{Q}_2 = (\mathbf{B} \quad \mathbf{0})$. Thus $\mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$ is a nonsingular matrix such that $\mathbf{AQ} = \mathbf{AQ}_1\mathbf{Q}_2 = (\mathbf{B} \quad \mathbf{N})\mathbf{Q}_2 = (\mathbf{B} \quad \mathbf{0})$, and hence $\mathbf{A} \overset{\text{col}}{\sim} (\mathbf{B} \quad \mathbf{0})$. The conclusion (4.2.8) now follows from (4.2.6). ∎

Example 4.2.3

Problem: Determine spanning sets for $R(\mathbf{A})$ and $R\left(\mathbf{A}^T\right)$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}.$$

Solution: Reducing $\mathbf{A}$ to any row echelon form $\mathbf{U}$ provides the solution—the basic columns in $\mathbf{A}$ correspond to the pivotal positions in $\mathbf{U}$, and the nonzero rows of $\mathbf{U}$ span the row space of $\mathbf{A}$. Using $\mathbf{E_A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ produces

$$R(\mathbf{A}) = span\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\} \quad \text{and} \quad R\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

So far, only two of the four fundamental subspaces associated with each matrix $\mathbf{A} \in \Re^{m\times n}$ have been discussed, namely, $R(\mathbf{A})$ and $R\left(\mathbf{A}^T\right)$. To see where the other two fundamental subspaces come from, consider again a general linear function f mapping $\Re^n$ into $\Re^m$, and focus on $\mathcal{N}(f) = \{\mathbf{x} \,|\, f(\mathbf{x}) = \mathbf{0}\}$ (the set of vectors that are mapped to $\mathbf{0}$). $\mathcal{N}(f)$ is called the ***nullspace*** of f (some texts call it the ***kernel*** of f), and it's easy to see that $\mathcal{N}(f)$ is a subspace of $\Re^n$ because the closure properties **(A1)** and **(M1)** are satisfied. Indeed, if $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{N}(f)$, then $f(\mathbf{x}_1) = \mathbf{0}$ and $f(\mathbf{x}_2) = \mathbf{0}$, so the linearity of f produces

$$f(\mathbf{x}_1 + \mathbf{x}_2) = f(\mathbf{x}_1) + f(\mathbf{x}_2) = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x}_1 + \mathbf{x}_2 \in \mathcal{N}(f). \quad \textbf{(A1)}$$

Similarly, if $\alpha \in \Re$, and if $\mathbf{x} \in \mathcal{N}(f)$, then $f(\mathbf{x}) = \mathbf{0}$ and linearity implies

$$f(\alpha\mathbf{x}) = \alpha f(\mathbf{x}) = \alpha\mathbf{0} = \mathbf{0} \implies \alpha\mathbf{x} \in \mathcal{N}(f). \quad \textbf{(M1)}$$

By considering the linear functions $f(\mathbf{x}) = \mathbf{Ax}$ and $g(\mathbf{y}) = \mathbf{A}^T\mathbf{y}$, the other two fundamental subspaces defined by $\mathbf{A} \in \Re^{m\times n}$ are obtained. They are $\mathcal{N}(f) = \{\mathbf{x}_{n\times 1} \mid \mathbf{Ax} = \mathbf{0}\} \subseteq \Re^n$ and $\mathcal{N}(g) = \left\{\mathbf{y}_{m\times 1} \mid \mathbf{A^T y} = \mathbf{0}\right\} \subseteq \Re^m$.