## Left-Hand Nullspace

If $rank(\mathbf{A}_{m\times n}) = r$, and if $\mathbf{PA} = \mathbf{U}$, where $\mathbf{P}$ is nonsingular and $\mathbf{U}$ is in row echelon form, then the last $m-r$ rows in $\mathbf{P}$ span the left-hand nullspace of $\mathbf{A}$. In other words, if $\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix}$, where $\mathbf{P}_2$ is $(m-r)\times m$, then

$$N\left(\mathbf{A}^T\right) = R\left(\mathbf{P}_2^T\right). \tag{4.2.12}$$

*Proof.* If $\mathbf{U} = \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \end{pmatrix}$, where $\mathbf{C}_{r\times n}$, then $\mathbf{PA} = \mathbf{U}$ implies $\mathbf{P}_2\mathbf{A} = \mathbf{0}$, and this says $R\left(\mathbf{P}_2^T\right) \subseteq N\left(\mathbf{A}^T\right)$. To show equality, demonstrate containment in the opposite direction by arguing that every vector in $N\left(\mathbf{A}^T\right)$ must also be in $R\left(\mathbf{P}_2^T\right)$. Suppose $\mathbf{y} \in N\left(\mathbf{A}^T\right)$, and let $\mathbf{P}^{-1} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{pmatrix}$ to conclude that

$$\mathbf{0} = \mathbf{y}^T\mathbf{A} = \mathbf{y}^T\mathbf{P}^{-1}\mathbf{U} = \mathbf{y}^T\mathbf{Q}_1\mathbf{C} \implies \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1$$

because $N\left(\mathbf{C}^T\right) = \{\mathbf{0}\}$ by (4.2.11). Now observe that $\mathbf{PP}^{-1} = \mathbf{I} = \mathbf{P}^{-1}\mathbf{P}$ insures $\mathbf{P}_1\mathbf{Q}_1 = \mathbf{I}_r$ and $\mathbf{Q}_1\mathbf{P}_1 = \mathbf{I}_m - \mathbf{Q}_2\mathbf{P}_2$, so

$$\begin{aligned} \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1 &\implies \mathbf{0} = \mathbf{y}^T\mathbf{Q}_1\mathbf{P}_1 = \mathbf{y}^T\left(\mathbf{I} - \mathbf{Q}_2\mathbf{P}_2\right) \\ &\implies \mathbf{y}^T = \mathbf{y}^T\mathbf{Q}_2\mathbf{P}_2 = \left(\mathbf{y}^T\mathbf{Q}_2\right)\mathbf{P}_2 \\ &\implies \mathbf{y} \in R\left(\mathbf{P}_2^T\right) \implies N\left(\mathbf{A}^T\right) \subseteq R\left(\mathbf{P}_2^T\right). \quad \blacksquare \end{aligned}$$

### Example 4.2.5

**Problem:** Determine a spanning set for $N\left(\mathbf{A}^T\right)$, where $\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{pmatrix}$.

**Solution:** To find a nonsingular matrix $\mathbf{P}$ such that $\mathbf{PA} = \mathbf{U}$ is in row echelon form, proceed as described in Exercise 3.9.1 and row reduce the augmented matrix $\left(\mathbf{A} \mid \mathbf{I}\right)$ to $\left(\mathbf{U} \mid \mathbf{P}\right)$. It must be the case that $\mathbf{PA} = \mathbf{U}$ because $\mathbf{P}$ is the product of the elementary matrices corresponding to the elementary row operations used. Since any row echelon form will suffice, we may use Gauss–Jordan reduction to reduce $\mathbf{A}$ to $\mathbf{E_A}$ as shown below:

$$\left(\begin{array}{cccc|ccc} 1 & 2 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 \\ 3 & 6 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|ccc} 1 & 2 & 0 & 1 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & 1 & 2/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/3 & -5/3 & 1 \end{array}\right)$$

$$\mathbf{P} = \begin{pmatrix} -1/3 & 2/3 & 0 \\ 2/3 & -1/3 & 0 \\ 1/3 & -5/3 & 1 \end{pmatrix}, \text{ so (4.2.12) implies } N\left(\mathbf{A}^T\right) = span\left\{\begin{pmatrix} 1/3 \\ -5/3 \\ 1 \end{pmatrix}\right\}.$$