is known as the ***triangle inequality.*** In higher-dimensional spaces we do not have the luxury of visualizing the geometry with our eyes, and the question of whether or not the triangle inequality remains valid has no obvious answer. The CBS inequality is precisely what is required to prove that, in this respect, the geometry of higher dimensions is no different than that of the visual spaces.

Triangle Inequality

$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\| \quad \text{for every } \mathbf{x},\, \mathbf{y} \in \mathcal{C}^n.$$

Proof. Consider $\mathbf{x}$ and $\mathbf{y}$ to be column vectors, and write

$$\begin{aligned} \|\mathbf{x}+\mathbf{y}\|^2 &= (\mathbf{x}+\mathbf{y})^*(\mathbf{x}+\mathbf{y}) = \mathbf{x}^*\mathbf{x} + \mathbf{x}^*\mathbf{y} + \mathbf{y}^*\mathbf{x} + \mathbf{y}^*\mathbf{y} \\ &= \|\mathbf{x}\|^2 + \mathbf{x}^*\mathbf{y} + \mathbf{y}^*\mathbf{x} + \|\mathbf{y}\|^2. \end{aligned} \tag{5.1.5}$$

Recall that if $z = a + \mathrm{i}b$, then $z + \bar{z} = 2a = 2\,\mathrm{Re}\,(z)$ and $|z|^2 = a^2 + b^2 \geq a^2$, so that $|z| \geq \mathrm{Re}\,(z)$. Using the fact that $\mathbf{y}^*\mathbf{x} = \overline{\mathbf{x}^*\mathbf{y}}$ together with the CBS inequality yields

$$\mathbf{x}^*\mathbf{y} + \mathbf{y}^*\mathbf{x} = 2\,\mathrm{Re}\,(\mathbf{x}^*\mathbf{y}) \leq 2\,|\mathbf{x}^*\mathbf{y}| \leq 2\,\|\mathbf{x}\|\;\|\mathbf{y}\|.$$

Consequently, we may infer from (5.1.5) that

$$\|\mathbf{x}+\mathbf{y}\|^2 \leq \|\mathbf{x}\|^2 + 2\,\|\mathbf{x}\|\;\|\mathbf{y}\| + \|\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \quad \blacksquare$$

It's not difficult to see that the triangle inequality can be extended to any number of vectors in the sense that $\left\|\sum_i \mathbf{x}_i\right\| \leq \sum_i \|\mathbf{x}_i\|$. Furthermore, it follows as a corollary that for real or complex numbers, $\left|\sum_i \alpha_i\right| \leq \sum_i |\alpha_i|$ (the triangle inequality for scalars).

Example 5.1.1

Backward Triangle Inequality. The triangle inequality produces an upper bound for a sum, but it also yields the following lower bound for a difference:

$$\big|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,\big| \leq \|\mathbf{x}-\mathbf{y}\|. \tag{5.1.6}$$

This is a consequence of the triangle inequality because

$$\|\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\| \implies \|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|$$

and

$$\|\mathbf{y}\| = \|\mathbf{x}-\mathbf{y}-\mathbf{x}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{x}\| \implies -(\|\mathbf{x}\| - \|\mathbf{y}\|) \leq \|\mathbf{x}-\mathbf{y}\|.$$