There are notions of length other than the euclidean measure. For example, urban dwellers navigate on a grid of city blocks with one-way streets, so they are prone to measure distances in the city not as the crow flies but rather in terms of lengths on a directed grid. For example, instead of than saying that "it's a one-half mile straight-line (euclidean) trip from here to there," they are more apt to describe the length of the trip by saying, "it's two blocks north on Dan Allen Drive, four blocks west on Hillsborough Street, and five blocks south on Gorman Street." In other words, the length of the trip is $2 + |-4| + |-5| = 11$ blocks—absolute value is used to insure that southerly and westerly movement does not cancel the effect of northerly and easterly movement, respectively. This "grid norm" is better known as the 1-norm because it is a special case of a more general class of norms defined below.

p-Norms

For $p \geq 1$, the ***p-norm*** of $\mathbf{x} \in \mathcal{C}^n$ is defined as $\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$.

It can be proven that the following properties of the euclidean norm are in fact valid for all p-norms:

$$\begin{aligned} &\|\mathbf{x}\|_p \geq 0 \quad \text{and} \quad \|\mathbf{x}\|_p = 0 \iff \mathbf{x} = \mathbf{0}, \\ &\|\alpha\mathbf{x}\|_p = |\alpha| \, \|\mathbf{x}\|_p \quad \text{for all scalars } \alpha, \\ &\|\mathbf{x}+\mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p \qquad \text{(see Exercise 5.1.13).} \end{aligned} \tag{5.1.7}$$

The generalized version of the CBS inequality (5.1.3) for p-norms is ***Hölder's inequality*** (developed in Exercise 5.1.12), which states that if $p > 1$ and $q > 1$ are real numbers such that $1/p + 1/q = 1$, then

$$|\mathbf{x}^*\mathbf{y}| \leq \|\mathbf{x}\|_p \, \|\mathbf{y}\|_q \, . \tag{5.1.8}$$

In practice, only three of the p-norms are used, and they are

$$\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i| \quad \text{(the grid norm)}, \quad \|\mathbf{x}\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2} \text{(the euclidean norm)},$$

and

$$\|\mathbf{x}\|_\infty = \lim_{p\to\infty} \|\mathbf{x}\|_p = \lim_{p\to\infty} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} = \max_i |x_i| \qquad \text{(the max norm)}.$$

For example, if $\mathbf{x} = (3,\ 4-3\mathrm{i},\ 1)$, then $\|\mathbf{x}\|_1 = 9$, $\|\mathbf{x}\|_2 = \sqrt{35}$, and $\|\mathbf{x}\|_\infty = 5$.