5.1.12. The classical form of ***Hölder's inequality***[36] states that if $p > 1$ and $q > 1$ are real numbers such that $1/p + 1/q = 1$, then

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q\right)^{1/q}.$$

Derive this inequality by executing the following steps:

(a) By considering the function $f(t) = (1-\lambda) + \lambda t - t^\lambda$ for $0 < \lambda < 1$, establish the inequality

$$\alpha^\lambda \beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$$

for nonnegative real numbers α and β.

(b) Let $\hat{\mathbf{x}} = \mathbf{x}/\left\|\mathbf{x}\right\|_p$ and $\hat{\mathbf{y}} = \mathbf{y}/\left\|\mathbf{y}\right\|_q$, and apply the inequality of part (a) to obtain

$$\sum_{i=1}^{n} |\hat{x}_i \hat{y}_i| \le \frac{1}{p}\sum_{i=1}^{n} |\hat{x}_i|^p + \frac{1}{q}\sum_{i=1}^{n} |\hat{y}_i|^q = 1.$$

(c) Deduce the classical form of Hölder's inequality, and then explain why this means that

$$|\mathbf{x}^*\mathbf{y}| \le \left\|\mathbf{x}\right\|_p \left\|\mathbf{y}\right\|_q.$$

5.1.13. The triangle inequality $\left\|\mathbf{x}+\mathbf{y}\right\|_p \le \left\|\mathbf{x}\right\|_p + \left\|\mathbf{y}\right\|_p$ for a general p-norm is really the classical ***Minkowski inequality,***[37] which states that for $p \ge 1$,

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1/p} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}.$$

Derive Minkowski's inequality. **Hint:** For $p > 1$, let q be the number such that $1/q = 1 - 1/p$. Verify that for scalars α and β,

$$|\alpha+\beta|^p = |\alpha+\beta|\,|\alpha+\beta|^{p/q} \le |\alpha|\,|\alpha+\beta|^{p/q} + |\beta|\,|\alpha+\beta|^{p/q},$$

and make use of Hölder's inequality in Exercise 5.1.12.

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[36] Ludwig Otto Hölder (1859–1937) was a German mathematician who studied at Göttingen and lived in Leipzig. Although he made several contributions to analysis as well as algebra, he is primarily known for the development of the inequality that now bears his name.

[37] Hermann Minkowski (1864–1909) was born in Russia, but spent most of his life in Germany as a mathematician and professor at Königsberg and Göttingen. In addition to the inequality that now bears his name, he is known for providing a mathematical basis for the special theory of relativity. He died suddenly from a ruptured appendix at the age of 44.