5.2 MATRIX NORMS

Because $\mathcal{C}^{m\times n}$ is a vector space of dimension mn, magnitudes of matrices $\mathbf{A} \in \mathcal{C}^{m\times n}$ can be "measured" by employing any vector norm on $\mathcal{C}^{mn}$. For example, by stringing out the entries of $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & -2 \end{pmatrix}$ into a four-component vector, the euclidean norm on $\Re^4$ can be applied to write

$$\|\mathbf{A}\| = \left[2^2 + (-1)^2 + (-4)^2 + (-2)^2\right]^{1/2} = 5.$$

This is one of the simplest notions of a matrix norm, and it is called the *Frobenius* (p. 662) *norm* (older texts refer to it as the *Hilbert–Schmidt norm* or the *Schur norm*). There are several useful ways to describe the Frobenius matrix norm.

Frobenius Matrix Norm

The ***Frobenius norm*** of $\mathbf{A} \in \mathcal{C}^{m\times n}$ is defined by the equations

$$\|\mathbf{A}\|_F^2 = \sum_{i,j} |a_{ij}|^2 = \sum_i \|\mathbf{A}_{i*}\|_2^2 = \sum_j \|\mathbf{A}_{*j}\|_2^2 = \mathit{trace}\,(\mathbf{A}^*\mathbf{A}). \qquad (5.2.1)$$

The Frobenius matrix norm is fine for some problems, but it is not well suited for all applications. So, similar to the situation for vector norms, alternatives need to be explored. But before trying to develop different recipes for matrix norms, it makes sense to first formulate a general definition of a matrix norm. The goal is to start with the defining properties for a vector norm given in (5.1.9) on p. 275 and ask what, if anything, needs to be added to that list.

Matrix multiplication distinguishes matrix spaces from more general vector spaces, but the three vector-norm properties (5.1.9) say nothing about products. So, an extra property that relates $\|\mathbf{AB}\|$ to $\|\mathbf{A}\|$ and $\|\mathbf{B}\|$ is needed. The Frobenius norm suggests the nature of this extra property. The CBS inequality insures that $\|\mathbf{Ax}\|_2^2 = \sum_i |\mathbf{A}_{i*}\mathbf{x}|^2 \le \sum_i \|\mathbf{A}_{i*}\|_2^2 \|\mathbf{x}\|_2^2 = \|\mathbf{A}\|_F^2 \|\mathbf{x}\|_2^2$. That is,

$$\|\mathbf{Ax}\|_2 \le \|\mathbf{A}\|_F \|\mathbf{x}\|_2, \qquad (5.2.2)$$

and we express this by saying that the Frobenius matrix norm $\|\star\|_F$ and the euclidean vector norm $\|\star\|_2$ are ***compatible***. The compatibility condition (5.2.2) implies that for all conformable matrices $\mathbf{A}$ and $\mathbf{B}$,

$$\|\mathbf{AB}\|_F^2 = \sum_j \|[\mathbf{AB}]_{*j}\|_2^2 = \sum_j \|\mathbf{AB}_{*j}\|_2^2 \le \sum_j \|\mathbf{A}\|_F^2 \|\mathbf{B}_{*j}\|_2^2$$
$$= \|\mathbf{A}\|_F^2 \sum_j \|\mathbf{B}_{*j}\|_2^2 = \|\mathbf{A}\|_F^2 \|\mathbf{B}\|_F^2 \implies \|\mathbf{AB}\|_F \le \|\mathbf{A}\|_F \|\mathbf{B}\|_F.$$

This suggests that the submultiplicative property $\|\mathbf{AB}\| \le \|\mathbf{A}\| \, \|\mathbf{B}\|$ should be added to (5.1.9) to define a general matrix norm.