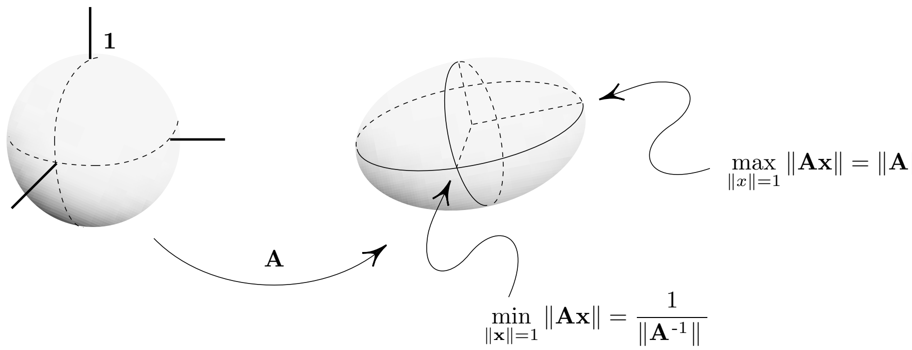

FIGURE 5.2.1. THE INDUCED MATRIX 2-NORM IN $\Re^3$.

Intuition might suggest that the euclidean vector norm should induce the Frobenius matrix norm (5.2.1), but something surprising happens instead.

Matrix 2-Norm

- The matrix norm induced by the euclidean vector norm is

$$\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{Ax}\|_2 = \sqrt{\lambda_{\max}}, \tag{5.2.7}$$

where $\lambda_{\max}$ is the largest number λ such that $\mathbf{A}^*\mathbf{A} - \lambda\mathbf{I}$ is singular.

- When $\mathbf{A}$ is nonsingular,

$$\left\|\mathbf{A}^{-1}\right\|_2 = \frac{1}{\displaystyle\min_{\|x\|_2=1} \|\mathbf{Ax}\|_2} = \frac{1}{\sqrt{\lambda_{\min}}}, \tag{5.2.8}$$

where $\lambda_{\min}$ is the smallest number λ such that $\mathbf{A}^*\mathbf{A} - \lambda\mathbf{I}$ is singular.

Note: If you are already familiar with eigenvalues, these say that $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues of $\mathbf{A}^*\mathbf{A}$ (Example 7.5.1, p. 549), while $(\lambda_{\max})^{1/2} = \sigma_1$ and $(\lambda_{\min})^{1/2} = \sigma_n$ are the largest and smallest singular values of $\mathbf{A}$ (p. 414).

Proof. To prove (5.2.7), assume that $\mathbf{A}_{m\times n}$ is real (a proof for complex matrices is given in Example 7.5.1 on p. 549). The strategy is to evaluate $\|\mathbf{A}\|_2^2$ by solving the problem

$$\text{maximize } f(\mathbf{x}) = \|\mathbf{Ax}\|_2^2 = \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} \quad \text{subject to } g(\mathbf{x}) = \mathbf{x}^T\mathbf{x} = 1$$