Norms in Inner-Product Spaces

If $\mathcal{V}$ is an inner-product space with an inner product $\langle \mathbf{x}|\mathbf{y}\rangle$, then

$$\|\star\| = \sqrt{\langle \star|\star\rangle} \quad \text{defines a norm on } \mathcal{V}.$$

Proof. The fact that $\|\star\| = \sqrt{\langle \star|\star\rangle}$ satisfies the first two norm properties in (5.2.3) on p. 280 follows directly from the defining properties (5.3.1) for an inner product. You are asked to provide the details in Exercise 5.3.3. To establish the triangle inequality, use $\langle \mathbf{x}|\mathbf{y}\rangle + \langle \mathbf{y}|\mathbf{x}\rangle = \langle \mathbf{x}|\mathbf{y}\rangle + \overline{\langle \mathbf{x}|\mathbf{y}\rangle} = 2\,\mathrm{Re}\,(\langle \mathbf{x}|\mathbf{y}\rangle) \le 2\,|\,\langle \mathbf{x}|\mathbf{y}\rangle\,|$ together with the CBS inequality to write

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|^2 &= \langle \mathbf{x}+\mathbf{y}|\mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}|\mathbf{x}\rangle + \langle \mathbf{x}|\mathbf{y}\rangle + \langle \mathbf{y}|\mathbf{x}\rangle + \langle \mathbf{y}|\mathbf{y}\rangle \\ &\le \|\mathbf{x}\|^2 + 2|\,\langle \mathbf{x}|\mathbf{y}\rangle\,| + \|\mathbf{y}\|^2 \le (\|\mathbf{x}\| + \|\mathbf{y}\|)^2. \quad \blacksquare\end{aligned}$$

Example 5.3.2

Problem: Describe the norms that are generated by the inner products presented in Example 5.3.1.

- Given a nonsingular matrix $\mathbf{A} \in \mathcal{C}^{n\times n}$, the $\mathbf{A}$***-norm*** (or ***elliptical norm***) generated by the $\mathbf{A}$-inner product on $\mathcal{C}^{n\times 1}$ is

$$\|\mathbf{x}\|_{\mathbf{A}} = \sqrt{\langle \mathbf{x}|\mathbf{x}\rangle} = \sqrt{\mathbf{x}^*\mathbf{A}^*\mathbf{A}\mathbf{x}} = \|\mathbf{A}\mathbf{x}\|_2\,. \tag{5.3.5}$$

- The standard inner product for matrices generates the Frobenius matrix norm because

$$\|\mathbf{A}\| = \sqrt{\langle \mathbf{A}|\mathbf{A}\rangle} = \sqrt{\mathit{trace}\,(\mathbf{A}^*\mathbf{A})} = \|\mathbf{A}\|_F\,. \tag{5.3.6}$$

- For the space of real-valued continuous functions defined on (a, b), the norm of a function f generated by the inner product $\langle f|g\rangle = \int_a^b f(t)g(t)dt$ is

$$\|f\| = \sqrt{\langle f|f\rangle} = \left(\int_a^b f(t)^2 dt\right)^{1/2}.$$