guarantees that $\langle \mathbf{x}|\mathbf{y}\rangle + \langle \mathbf{x}|\mathbf{z}\rangle = \langle \mathbf{x}|\mathbf{y}+\mathbf{z}\rangle$. Now prove that $\langle \mathbf{x}|\alpha\mathbf{y}\rangle = \alpha\langle \mathbf{x}|\mathbf{y}\rangle$ for all real α. This is valid for integer values of α by the result just established, and it holds when α is rational because if β and γ are integers, then

$$\gamma^2 \left\langle \mathbf{x} \middle| \frac{\beta}{\gamma}\mathbf{y} \right\rangle = \langle \gamma\mathbf{x}|\beta\mathbf{y}\rangle = \beta\gamma\langle \mathbf{x}|\mathbf{y}\rangle \implies \left\langle \mathbf{x} \middle| \frac{\beta}{\gamma}\mathbf{y} \right\rangle = \frac{\beta}{\gamma}\langle \mathbf{x}|\mathbf{y}\rangle.$$

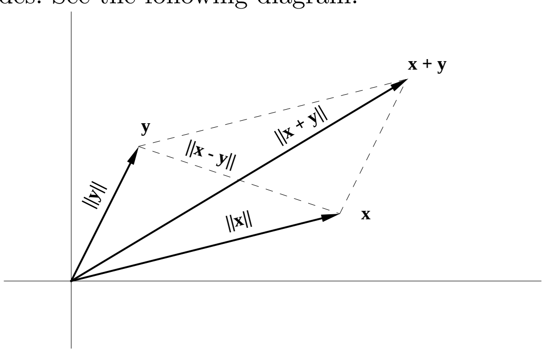
Because $\|\mathbf{x}+\alpha\mathbf{y}\|$ and $\|\mathbf{x}-\alpha\mathbf{y}\|$ are continuous functions of α (Exercise 5.1.7), equation (5.3.8) insures that $\langle \mathbf{x}|\alpha\mathbf{y}\rangle$ is a continuous function of α. Therefore, if α is irrational, and if $\{\alpha_n\}$ is a sequence of rational numbers such that $\alpha_n \to \alpha$, then $\langle \mathbf{x}|\alpha_n\mathbf{y}\rangle \to \langle \mathbf{x}|\alpha\mathbf{y}\rangle$ and $\langle \mathbf{x}|\alpha_n\mathbf{y}\rangle = \alpha_n\langle \mathbf{x}|\mathbf{y}\rangle \to \alpha\langle \mathbf{x}|\mathbf{y}\rangle$, so $\langle \mathbf{x}|\alpha\mathbf{y}\rangle = \alpha\langle \mathbf{x}|\mathbf{y}\rangle$. ■

Example 5.3.4

We already know that the euclidean vector norm on $\mathcal{C}^n$ is generated by the standard inner product, so the previous theorem guarantees that the parallelogram identity must hold for the 2-norm. This is easily corroborated by observing that

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_2^2 + \|\mathbf{x}-\mathbf{y}\|_2^2 &= (\mathbf{x}+\mathbf{y})^*(\mathbf{x}+\mathbf{y}) + (\mathbf{x}-\mathbf{y})^*(\mathbf{x}-\mathbf{y}) \\ &= 2\left(\mathbf{x}^*\mathbf{x}+\mathbf{y}^*\mathbf{y}\right) = 2(\|\mathbf{x}\|_2^2 + \|\mathbf{y}\|_2^2).\end{aligned}$$

The parallelogram identity is so named because it expresses the fact that the sum of the squares of the diagonals in a parallelogram is twice the sum of the squares of the sides. See the following diagram.



Example 5.3.5

Problem: Except for the euclidean norm, is any other vector p-norm generated by an inner product?

Solution: No, because the parallelogram identity (5.3.7) doesn't hold when $p \neq 2$. To see that $\|\mathbf{x}+\mathbf{y}\|_p^2 + \|\mathbf{x}-\mathbf{y}\|_p^2 = 2\big(\|\mathbf{x}\|_p^2 + \|\mathbf{y}\|_p^2\big)$ is not valid for all $\mathbf{x},\mathbf{y} \in \mathcal{C}^n$ when $p \neq 2$, consider $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$. It's apparent that $\|\mathbf{e_1}+\mathbf{e_2}\|_p^2 = 2^{2/p} = \|\mathbf{e_1}-\mathbf{e_2}\|_p^2$, so

$$\|\mathbf{e_1}+\mathbf{e_2}\|_p^2 + \|\mathbf{e_1}-\mathbf{e_2}\|_p^2 = 2^{(p+2)/p} \quad \text{and} \quad 2\big(\|\mathbf{e_1}\|_p^2 + \|\mathbf{e_2}\|_p^2\big) = 4.$$