Furthermore, the fact that

$$\xi^k \left(1 + \xi^k + \xi^{2k} + \cdots + \xi^{(n-2)k} + \xi^{(n-1)k}\right) = \xi^k + \xi^{2k} + \cdots + \xi^{(n-1)k} + 1$$

implies $\left(1 + \xi^k + \xi^{2k} + \cdots + \xi^{(n-1)k}\right)\left(1 - \xi^k\right) = 0$ and, consequently,

$$1 + \xi^k + \xi^{2k} + \cdots + \xi^{(n-1)k} = 0 \quad \text{whenever} \quad \xi^k \neq 1. \tag{5.8.2}$$

Fourier Matrix

The $n \times n$ matrix whose (j,k)-entry is $\xi^{jk} = \omega^{-jk}$ for $0 \leq j, k \leq n-1$ is called the ***Fourier matrix*** of order n, and it has the form

$$\mathbf{F}_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{n-2} & \cdots & \xi \end{pmatrix}_{n \times n}.$$

Note. Throughout this section entries are indexed from 0 to $n-1$. For example, the upper left-hand entry of $\mathbf{F}_n$ is considered to be in the $(0,0)$ position (rather than the $(1,1)$ position), and the lower right-hand entry is in the $(n-1, n-1)$ position. When the context makes it clear, the subscript n on $\mathbf{F}_n$ is omitted.

The Fourier matrix[50] is a special case of the Vandermonde matrix introduced in Example 4.3.4. Using (5.8.1) and (5.8.2), we see that the inner product of any two columns in $\mathbf{F}_n$, say, the r^{th} and s^{th}, is

$$\mathbf{F}_{*r}^* \mathbf{F}_{*s} = \sum_{j=0}^{n-1} \overline{\xi^{jr}} \xi^{js} = \sum_{j=0}^{n-1} \xi^{-jr} \xi^{js} = \sum_{j=0}^{n-1} \xi^{j(s-r)} = 0.$$

In other words, the columns in $\mathbf{F}_n$ are mutually orthogonal. Furthermore, each column in $\mathbf{F}_n$ has norm $\sqrt{n}$ because

$$\|\mathbf{F}_{*k}\|_2^2 = \sum_{j=0}^{n-1} |\xi^{jk}|^2 = \sum_{j=0}^{n-1} 1 = n,$$

[50] Some authors define the Fourier matrix using powers of ω rather than powers of ξ, and some include a scalar multiple $1/n$ or $1/\sqrt{n}$. These differences are superficial, and they do not affect the basic properties. Our definition is the discrete counterpart of the integral operator $F(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$ that is usually taken as the definition of the continuous Fourier transform.