Fast Fourier Transform

For a given input vector $\mathbf{x}$ containing $n = 2^r$ components, the discrete Fourier transform $\mathbf{F}_n\mathbf{x}$ is the result of successively creating the following arrays.

$\mathbf{X}_{1\times n} \longleftarrow rev(\mathbf{x})$ (bit reverse the subscripts)

For $j = 0, 1, 2, 3, \ldots, r-1$

$$\mathbf{D} \longleftarrow \begin{pmatrix} 1 \\ e^{-\pi i/2^j} \\ e^{-2\pi i/2^j} \\ e^{-3\pi i/2^j} \\ \vdots \\ e^{-(2^j-1)\pi i/2^j} \end{pmatrix}_{2^j\times 1} \quad \text{(Half of the } (2^{j+1})^{th} \text{ roots of 1, perhaps from a lookup table)}$$

$$\mathbf{X}^{(0)} \longleftarrow \begin{pmatrix} \mathbf{X}_{*0} & \mathbf{X}_{*2} & \mathbf{X}_{*4} & \cdots & \mathbf{X}_{*2^{r-j}-2} \end{pmatrix}_{2^j\times 2^{r-j-1}}$$

$$\mathbf{X}^{(1)} \longleftarrow \begin{pmatrix} \mathbf{X}_{*1} & \mathbf{X}_{*3} & \mathbf{X}_{*5} & \cdots & \mathbf{X}_{*2^{r-j}-1} \end{pmatrix}_{2^j\times 2^{r-j-1}}$$

$$\mathbf{X} \longleftarrow \begin{pmatrix} \mathbf{X}^{(0)} + \mathbf{D}\times\mathbf{X}^{(1)} \\ \mathbf{X}^{(0)} - \mathbf{D}\times\mathbf{X}^{(1)} \end{pmatrix}_{2^{j+1}\times 2^{r-j-1}} \quad \begin{pmatrix} \text{Define } \times \text{ to mean} \\ [\mathbf{D}\times\mathbf{M}]_{ij} = d_i m_{ij} \end{pmatrix}$$

Example 5.8.5

Problem: Perform the FFT on $\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Solution: Start with $\mathbf{X} \longleftarrow rev(\mathbf{x}) = \begin{pmatrix} x_0 & x_2 & x_1 & x_3 \end{pmatrix}$.

For $j = 0$:

$\mathbf{D} \longleftarrow (1)$ (Half of the square roots of 1)

$\mathbf{X}^{(0)} \longleftarrow \begin{pmatrix} x_0 & x_1 \end{pmatrix}$

$\mathbf{X}^{(1)} \longleftarrow \begin{pmatrix} x_2 & x_3 \end{pmatrix}$ and $\mathbf{D}\times\mathbf{X}^{(1)} \longleftarrow \begin{pmatrix} x_2 & x_3 \end{pmatrix}$

$$\mathbf{X} \longleftarrow \begin{pmatrix} \mathbf{X}^{(0)} + \mathbf{D}\times\mathbf{X}^{(1)} \\ \mathbf{X}^{(0)} - \mathbf{D}\times\mathbf{X}^{(1)} \end{pmatrix} = \begin{pmatrix} x_0 + x_2 & x_1 + x_3 \\ x_0 - x_2 & x_1 - x_3 \end{pmatrix}$$