5.9 COMPLEMENTARY SUBSPACES

The sum of two subspaces $\mathcal{X}$ and $\mathcal{Y}$ of a vector space $\mathcal{V}$ was defined on p. 166 to be the set $\mathcal{X}+\mathcal{Y}=\{\mathbf{x}+\mathbf{y} \mid \mathbf{x}\in\mathcal{X} \text{ and } \mathbf{y}\in\mathcal{Y}\}$, and it was established that $\mathcal{X}+\mathcal{Y}$ is another subspace of $\mathcal{V}$. For example, consider the two subspaces of $\Re^3$ shown in Figure 5.9.1 in which $\mathcal{X}$ is a plane through the origin, and $\mathcal{Y}$ is a line through the origin.

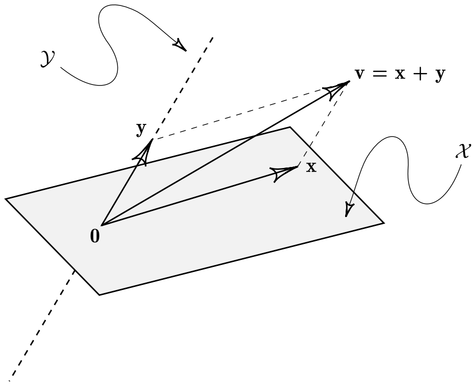

Figure 5.9.1

Notice that $\mathcal{X}$ and $\mathcal{Y}$ are ***disjoint*** in the sense that $\mathcal{X}\cap\mathcal{Y}=\mathbf{0}$. The parallelogram law for vector addition makes it clear that $\mathcal{X}+\mathcal{Y}=\Re^3$ because each vector in $\Re^3$ can be written as "something from $\mathcal{X}$ plus something from $\mathcal{Y}$." Thus $\Re^3$ is resolved into a pair of disjoint components $\mathcal{X}$ and $\mathcal{Y}$. These ideas generalize as described below.

Complementary Subspaces

Subspaces $\mathcal{X}$, $\mathcal{Y}$ of a space $\mathcal{V}$ are said to be ***complementary*** whenever

$$\mathcal{V}=\mathcal{X}+\mathcal{Y} \quad \text{and} \quad \mathcal{X}\cap\mathcal{Y}=\mathbf{0}, \tag{5.9.1}$$

in which case $\mathcal{V}$ is said to be the ***direct sum*** of $\mathcal{X}$ and $\mathcal{Y}$, and this is denoted by writing $\mathcal{V}=\mathcal{X}\oplus\mathcal{Y}$.

- For a vector space $\mathcal{V}$ with subspaces $\mathcal{X},\mathcal{Y}$ having respective bases $\mathcal{B}_{\mathcal{X}}$ and $\mathcal{B}_{\mathcal{Y}}$, the following statements are equivalent.
  - ▷ $\mathcal{V}=\mathcal{X}\oplus\mathcal{Y}$. (5.9.2)
  - ▷ For each $\mathbf{v}\in\mathcal{V}$ there are *unique* vectors $\mathbf{x}\in\mathcal{X}$ and $\mathbf{y}\in\mathcal{Y}$ such that $\mathbf{v}=\mathbf{x}+\mathbf{y}$. (5.9.3)
  - ▷ $\mathcal{B}_{\mathcal{X}}\cap\mathcal{B}_{\mathcal{Y}}=\phi$ (empty set) and $\mathcal{B}_{\mathcal{X}}\cup\mathcal{B}_{\mathcal{Y}}$ is a basis for $\mathcal{V}$. (5.9.4)