Example 5.9.2

Angle between Complementary Subspaces. The angle between nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in $\Re^n$ was defined on p. 295 to be the number $0 \leq \theta \leq \pi/2$ such that $\cos\theta = \mathbf{v}^T\mathbf{u}/\left\|\mathbf{v}\right\|_2\left\|\mathbf{u}\right\|_2$. It's natural to try to extend this idea to somehow make sense of angles between subspaces of $\Re^n$. Angles between completely general subspaces are presently out of our reach—they are discussed in §5.15—but the angle between a pair of *complementary* subspaces is within our grasp. When $\Re^n = \mathcal{R} \oplus \mathcal{N}$ with $\mathcal{R} \neq \mathbf{0} \neq \mathcal{N}$, the ***angle*** (also known as the ***minimal angle***) between $\mathcal{R}$ and $\mathcal{N}$ is defined to be the number $0 < \theta \leq \pi/2$ that satisfies

$$\cos\theta = \max_{\substack{\mathbf{u}\in\mathcal{R}\\ \mathbf{v}\in\mathcal{N}}} \frac{\mathbf{v}^T\mathbf{u}}{\left\|\mathbf{v}\right\|_2\left\|\mathbf{u}\right\|_2} = \max_{\substack{\mathbf{u}\in\mathcal{R},\,\mathbf{v}\in\mathcal{N}\\ \left\|\mathbf{u}\right\|_2=\left\|\mathbf{v}\right\|_2=1}} \mathbf{v}^T\mathbf{u}. \tag{5.9.16}$$

While this is a good definition, it's not easy to use—especially if one wants to compute the numerical value of $\cos\theta$. The trick in making θ more accessible is to think in terms of projections and $\sin\theta = (1-\cos^2\theta)^{1/2}$. Let $\mathbf{P}$ be the projector such that $R(\mathbf{P}) = \mathcal{R}$ and $N(\mathbf{P}) = \mathcal{N}$, and recall that the matrix 2-norm (p. 281) of $\mathbf{P}$ is

$$\left\|\mathbf{P}\right\|_2 = \max_{\left\|\mathbf{x}\right\|_2=1}\left\|\mathbf{P}\mathbf{x}\right\|_2. \tag{5.9.17}$$

In other words, $\left\|\mathbf{P}\right\|_2$ is the length of a longest vector in the image of the unit sphere under transformation by $\mathbf{P}$. To understand how $\sin\theta$ is related to $\left\|\mathbf{P}\right\|_2$, consider the situation in $\Re^3$. The image of the unit sphere under $\mathbf{P}$ is obtained by projecting the sphere onto $\mathcal{R}$ along lines parallel to $\mathcal{N}$. As depicted in Figure 5.9.2, the result is an ellipse in $\mathcal{R}$.

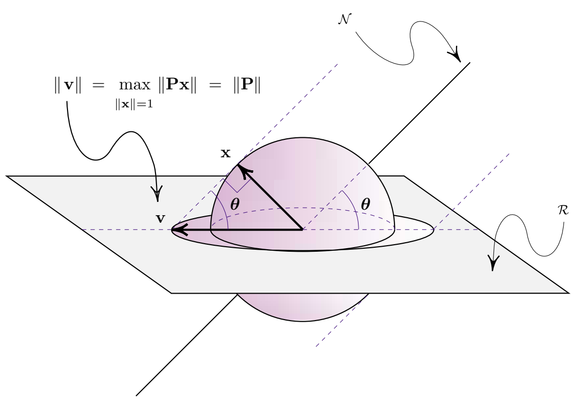

FIGURE 5.9.2

The norm of a longest vector $\mathbf{v}$ on this ellipse equals the norm of $\mathbf{P}$. That is, $\left\|\mathbf{v}\right\|_2 = \max_{\left\|\mathbf{x}\right\|_2=1}\left\|\mathbf{P}\mathbf{x}\right\|_2 = \left\|\mathbf{P}\right\|_2$, and it is apparent from the right triangle in