holds, and this forces $n < \dim N\left(\mathbf{A}^{n+1}\right)$, which is impossible. A similar argument proves equality exists somewhere in the range chain. ■

Property 2. *Once equality is attained, it is maintained throughout the rest of both chains in* (5.10.2). *In other words,*

$$\begin{aligned} N\left(\mathbf{A}^0\right) \subset N\left(\mathbf{A}\right) \subset \cdots \subset N\left(\mathbf{A}^k\right) = N\left(\mathbf{A}^{k+1}\right) = N\left(\mathbf{A}^{k+2}\right) = \cdots \\ R\left(\mathbf{A}^0\right) \supset R\left(\mathbf{A}\right) \supset \cdots \supset R\left(\mathbf{A}^k\right) = R\left(\mathbf{A}^{k+1}\right) = R\left(\mathbf{A}^{k+2}\right) = \cdots. \end{aligned} \tag{5.10.3}$$

Proof. If $k \geq 0$ is the smallest integer such that $R\left(\mathbf{A}^k\right) = R\left(\mathbf{A}^{k+1}\right)$, then $R\left(\mathbf{A}^{i+k}\right) = R\left(\mathbf{A}^i\mathbf{A}^k\right) = \mathbf{A}^i R\left(\mathbf{A}^k\right) = \mathbf{A}^i R\left(\mathbf{A}^{k+1}\right) = R\left(\mathbf{A}^{i+k+1}\right)$. The rank plus nullity theorem (p. 199) insures that the nullspaces stop growing at the same place the ranges stop shrinking. ■

Property 3. *Let* k *be the value at which the ranges stop shrinking and the nullspaces stop growing in* (5.10.3). *For a singular* $\mathbf{A}_{n\times n}$ *and an integer* $p > 0$,

$$R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) = \mathbf{0} \Longleftrightarrow R\left(\mathbf{A}^p\right) \oplus N\left(\mathbf{A}^p\right) = \Re^n \Longleftrightarrow p \geq k.$$

Proof. If $R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) = \mathbf{0}$, use (4.4.19), (4.4.15), and (4.4.6) to write

$$\begin{aligned} \dim\left[R\left(\mathbf{A}^p\right) + N\left(\mathbf{A}^p\right)\right] &= \dim R\left(\mathbf{A}^p\right) + \dim N\left(\mathbf{A}^p\right) - \dim R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) \\ &= \dim R\left(\mathbf{A}^p\right) + \dim N\left(\mathbf{A}^p\right) = n \quad \Longrightarrow \quad R\left(\mathbf{A}^p\right) + N\left(\mathbf{A}^p\right) = \Re^n. \end{aligned}$$

Consequently, $R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) = \mathbf{0}$ if and only if $R\left(\mathbf{A}^p\right) \oplus N\left(\mathbf{A}^p\right) = \Re^n$. Now prove $R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) = \mathbf{0} \Longleftrightarrow p \geq k$. Suppose $p \geq k$. If $\mathbf{x} \in R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right)$, then $\mathbf{A}^p\mathbf{y} = \mathbf{x}$ for some $\mathbf{y} \in \Re^n$, and $\mathbf{A}^p\mathbf{x} = \mathbf{0}$, so $\mathbf{A}^{2p}\mathbf{y} = \mathbf{A}^p\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{y} \in N\left(\mathbf{A}^{2p}\right) = N\left(\mathbf{A}^p\right) \Rightarrow \mathbf{x} = \mathbf{0} \Rightarrow R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) = \mathbf{0}$. Conversely, if $R\left(\mathbf{A}^p\right) \cap N\left(\mathbf{A}^p\right) = \mathbf{0}$, then $R\left(\mathbf{A}^p\right) \oplus N\left(\mathbf{A}^p\right) = \Re^n$, so $R\left(\mathbf{A}^p\right) = \mathbf{A}^p(\Re^n) = \mathbf{A}^p(R\left(\mathbf{A}^p\right)) = R\left(\mathbf{A}^{2p}\right) \Rightarrow p \geq k$, for otherwise $rank\left(\mathbf{A}^{p+1}\right) < rank\left(\mathbf{A}^p\right)$ (by Property 2), which would mean that $rank\left(\mathbf{A}^{2p}\right) < rank\left(\mathbf{A}^p\right)$.

Below is a summary of our observations concerning the index of a matrix.

Index

The index of a square matrix $\mathbf{A}$ is the smallest nonnegative integer k such that any one of the three following statements is true.

- $rank\left(\mathbf{A}^k\right) = rank\left(\mathbf{A}^{k+1}\right)$.
- $R\left(\mathbf{A}^k\right) = R\left(\mathbf{A}^{k+1}\right)$—i.e., the point where $R\left(\mathbf{A}^k\right)$ stops shrinking.
- $N\left(\mathbf{A}^k\right) = N\left(\mathbf{A}^{k+1}\right)$—i.e., the point where $N\left(\mathbf{A}^k\right)$ stops growing.

For nonsingular matrices, $index(\mathbf{A}) = 0$. For singular matrices, $index(\mathbf{A})$ is the smallest positive integer k such that either of the following two statements is true.

- $R\left(\mathbf{A}^k\right) \cap N\left(\mathbf{A}^k\right) = \mathbf{0}$. (5.10.4)
- $\Re^n = R\left(\mathbf{A}^k\right) \oplus N\left(\mathbf{A}^k\right)$.