and write

$$\begin{pmatrix} \mathbf{C}^k & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^k \end{pmatrix} = \mathbf{Q}^{-1}\mathbf{A}^k\mathbf{Q} = \begin{pmatrix} \mathbf{U} \\ \hline \mathbf{V} \end{pmatrix} \mathbf{A}^k (\mathbf{X} \,|\, \mathbf{Y}) = \begin{pmatrix} \mathbf{U}\mathbf{A}^k\mathbf{X} & \mathbf{0} \\ \mathbf{V}\mathbf{A}^k\mathbf{X} & \mathbf{0} \end{pmatrix}.$$

Therefore, $\mathbf{N}^k = \mathbf{0}$ and $\mathbf{Q}^{-1}\mathbf{A}^k\mathbf{Q} = \begin{pmatrix} \mathbf{C}^k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Since $\mathbf{C}^k$ is $r \times r$ and $r = rank\left(\mathbf{A}^k\right) = rank\left(\mathbf{Q}^{-1}\mathbf{A}^k\mathbf{Q}\right) = rank\left(\mathbf{C}^k\right)$, it must be the case that $\mathbf{C}^k$ is nonsingular, and hence $\mathbf{C}$ is nonsingular. Finally, notice that $index(\mathbf{N}) = k$ because if $index(\mathbf{N}) \neq k$, then $\mathbf{N}^{k-1} = \mathbf{0}$, so

$$rank\left(\mathbf{A}^{k-1}\right) = rank\left(\mathbf{Q}^{-1}\mathbf{A}^{k-1}\mathbf{Q}\right) = rank\begin{pmatrix} \mathbf{C}^{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}^{k-1} \end{pmatrix} = rank\begin{pmatrix} \mathbf{C}^{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$
$$= rank\left(\mathbf{C}^{k-1}\right) = r = rank\left(\mathbf{A}^k\right),$$

which is impossible because $index(\mathbf{A}) = k$ is the smallest integer for which there is equality in ranks of powers. ∎

Example 5.10.3

Problem: Let $\mathbf{A}_{n\times n}$ have index k with $rank\left(\mathbf{A}^k\right) = r$, and let

$$\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \begin{pmatrix} \mathbf{C}_{r\times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix} \quad \text{with} \quad \mathbf{Q} = \left(\mathbf{X}_{n\times r} \,|\, \mathbf{Y}\right) \text{ and } \mathbf{Q}^{-1} = \begin{pmatrix} \mathbf{U}_{r\times n} \\ \hline \mathbf{V} \end{pmatrix}$$

be the core-nilpotent decomposition described in (5.10.5). Explain why

$$\mathbf{Q}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{Q}^{-1} = \mathbf{X}\mathbf{U} = \text{the projector onto } R\left(\mathbf{A}^k\right) \text{ along } N\left(\mathbf{A}^k\right)$$

and

$$\mathbf{Q}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{Q}^{-1} = \mathbf{Y}\mathbf{V} = \text{the projector onto } N\left(\mathbf{A}^k\right) \text{ along } R\left(\mathbf{A}^k\right).$$

Solution: Because $R\left(\mathbf{A}^k\right)$ and $N\left(\mathbf{A}^k\right)$ are complementary subspaces, and because the columns of $\mathbf{X}$ and $\mathbf{Y}$ constitute respective bases for these spaces, it follows from the discussion concerning projectors on p. 386 that

$$\mathbf{P} = (\mathbf{X} \,|\, \mathbf{Y})\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}(\mathbf{X} \,|\, \mathbf{Y})^{-1} = \mathbf{Q}\begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{Q}^{-1} = \mathbf{X}\mathbf{U}$$

must be the projector onto $R\left(\mathbf{A}^k\right)$ along $N\left(\mathbf{A}^k\right)$, and

$$\mathbf{I} - \mathbf{P} = (\mathbf{X} \,|\, \mathbf{Y})\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}(\mathbf{X} \,|\, \mathbf{Y})^{-1} = \mathbf{Q}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{n-r} \end{pmatrix}\mathbf{Q}^{-1} = \mathbf{Y}\mathbf{V}$$

is the complementary projector onto $N\left(\mathbf{A}^k\right)$ along $R\left(\mathbf{A}^k\right)$.