Therefore,

$$\left\{\begin{array}{lcl} z_k & = & \alpha_k \cos\left(t\sqrt{\lambda_k}\right) + \beta_k \sin\left(t\sqrt{\lambda_k}\right) \\ z_k(0) & = & \tilde{c}_k \\ z_k'(0) & = & 0 \end{array}\right\} \implies z_k = \tilde{c}_k \cos\left(t\sqrt{\lambda_k}\right), \quad (7.6.3)$$

and for $\mathbf{P} = \left[\mathbf{x}_1 \,|\, \mathbf{x}_2 \,|\cdots|\, \mathbf{x}_n\right]$,

$$\mathbf{y} = \mathbf{Pz} = z_1\mathbf{x}_1 + z_2\mathbf{x}_2 + \cdots + z_n\mathbf{x}_n = \sum_{j=1}^{n}\left(\tilde{c}_j \cos\left(t\sqrt{\lambda_k}\right)\right)\mathbf{x}_j. \quad (7.6.4)$$

This means that every possible mode of vibration is a combination of modes determined by the eigenvectors $\mathbf{x}_j$. To understand this more clearly, suppose that the beads are initially positioned according to the components of $\mathbf{x}_j$ —i.e., $\mathbf{c} = \mathbf{y}(0) = \mathbf{x}_j$. Then $\tilde{\mathbf{c}} = \mathbf{P}^T\mathbf{c} = \mathbf{P}^T\mathbf{x}_j = \mathbf{e}_j$, so (7.6.3) and (7.6.4) reduce to

$$z_k = \begin{cases} \cos\left(t\sqrt{\lambda_k}\right) & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases} \implies \mathbf{y} = \left(\cos\left(t\sqrt{\lambda_k}\right)\right)\mathbf{x}_j. \quad (7.6.5)$$

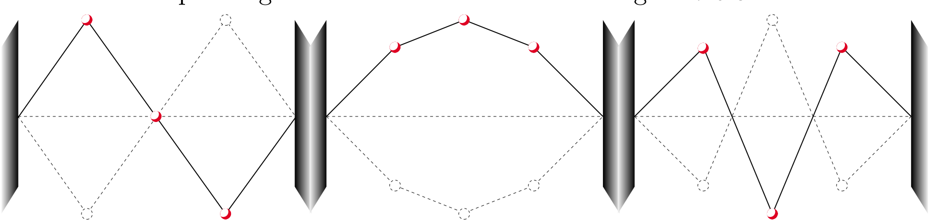
In other words, when $\mathbf{y}(0) = \mathbf{x}_j$, the j^{th} eigenpair $(\lambda_j, \mathbf{x}_j)$ completely determines the mode of vibration because the amplitudes are determined by $\mathbf{x}_j$, and each bead vibrates with a common frequency $f = \sqrt{\lambda_j}/2\pi$. This type of motion (7.6.5) is called a ***normal mode of vibration***. In these terms, equation (7.6.4) translates to say that *every possible mode of vibration is a combination of the normal modes*. For example, when $n = 3$, the matrix in (7.6.2) is

$$\mathbf{A} = \frac{T}{mL}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \quad \text{with} \quad \left\{\begin{array}{l} \lambda_1 = (T/mL)(2) \\ \lambda_2 = (T/mL)(2-\sqrt{2}) \\ \lambda_3 = (T/mL)(2+\sqrt{2}) \end{array}\right\},$$

and a complete orthonormal set of eigenvectors is

$$\mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}, \quad \mathbf{x}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}.$$

The three corresponding normal modes are shown in Figure 7.6.3.



Mode for $(\lambda_1, \mathbf{x}_1)$ Mode for $(\lambda_2, \mathbf{x}_2)$ Mode for $(\lambda_3, \mathbf{x}_3)$

FIGURE 7.6.3