Example 7.6.5

Taylor's theorem in $\Re^n$ says that if f is a smooth real-valued function defined on $\Re^n$, and if $\mathbf{x}_0 \in \Re^{n\times 1}$, then the value of f at $\mathbf{x} \in \Re^{n\times 1}$ is given by

$$f(\mathbf{x}) = f(\mathbf{x}_0) + (\mathbf{x}-\mathbf{x}_0)^T\mathbf{g}(\mathbf{x}_0) + \frac{(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)}{2} + O(\|\mathbf{x}-\mathbf{x}_0\|^3),$$

where $\mathbf{g}(\mathbf{x}_0) = \nabla f(\mathbf{x}_0)$ (the gradient of f evaluated at $\mathbf{x}_0$) has components $g_i = \partial f/\partial x_i\Big|_{\mathbf{x}_0}$, and where $\mathbf{H}(\mathbf{x}_0)$ is the ***Hessian matrix*** whose entries are given by $h_{ij} = \partial^2 f/\partial x_i \partial x_j\Big|_{\mathbf{x}_0}$. Just as in the case of one variable, the vector $\mathbf{x}_0$ is called a *critical point* when $\mathbf{g}(\mathbf{x}_0) = \mathbf{0}$. If $\mathbf{x}_0$ is a critical point, then Taylor's theorem shows that $(\mathbf{x}-\mathbf{x}_0)^T\mathbf{H}(\mathbf{x}_0)(\mathbf{x}-\mathbf{x}_0)$ governs the behavior of f at points $\mathbf{x}$ near to $\mathbf{x}_0$. This observation yields the following conclusions regarding local maxima or minima.

- If $\mathbf{x}_0$ is a critical point such that $\mathbf{H}(\mathbf{x}_0)$ is positive definite, then f has a local minimum at $\mathbf{x}_0$.
- If $\mathbf{x}_0$ is a critical point such that $\mathbf{H}(\mathbf{x}_0)$ is *negative definite* (i.e., $\mathbf{z}^T\mathbf{H}\mathbf{z} < 0$ for all $\mathbf{z} \neq \mathbf{0}$ or, equivalently, $-\mathbf{H}$ is positive definite), then f has a local maximum at $\mathbf{x}_0$.

Exercises for section 7.6

7.6.1. Which of the following matrices are positive definite?

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 5 & 1 \\ -1 & 1 & 5 \end{pmatrix}. \quad \mathbf{B} = \begin{pmatrix} 20 & 6 & 8 \\ 6 & 3 & 0 \\ 8 & 0 & 8 \end{pmatrix}. \quad \mathbf{C} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 6 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

7.6.2. Spring-Mass Vibrations. Two masses m_1 and m_2 are suspended between three identical springs (with spring constant k) as shown in Figure 7.6.7. Each mass is initially displaced from its equilibrium position by a horizontal distance and released to vibrate freely (assume there is no vertical displacement).

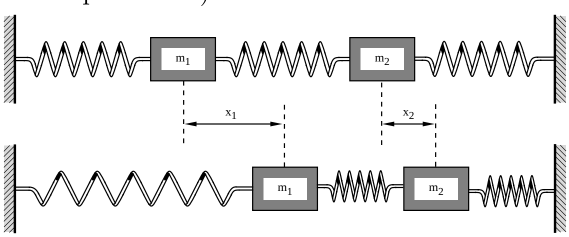

FIGURE 7.6.7