(a) If $x_i(t)$ denotes the horizontal displacement of m_i from equilibrium at time t, show that $\mathbf{Mx}'' = \mathbf{Kx}$, where

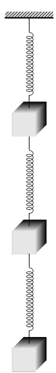
$$\mathbf{M} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{K} = k\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

(Consider a force directed to the left to be positive.) Notice that the *mass-stiffness equation* $\mathbf{Mx}'' = \mathbf{Kx}$ is the matrix version of Hooke's law $F = kx$, and $\mathbf{K}$ is positive definite.

(b) Look for a solution of the form $\mathbf{x} = e^{i\theta t}\mathbf{v}$ for a constant vector $\mathbf{v}$, and show that this reduces the problem to solving an algebraic equation of the form $\mathbf{Kv} = \lambda\mathbf{Mv}$ (for $\lambda = -\theta^2$). This is called a ***generalized eigenvalue problem*** because when $\mathbf{M} = \mathbf{I}$ we are back to the ordinary eigenvalue problem. The *generalized eigenvalues* λ_1 and λ_2 are the roots of the equation $\det(\mathbf{K} - \lambda\mathbf{M}) = 0$—find them when $k = 1$, $m_1 = 1$, and $m_2 = 2$, and describe the two modes of vibration.

(c) Take $m_1 = m_2 = m$, and apply the technique used in the vibrating beads problem in Example 7.6.1 (p. 559) to determine the normal modes. Compare the results with those of part (b).

7.6.3. Three masses m_1, m_2, and m_3 are suspended on three identical springs (with spring constant k) as shown below. Each mass is initially displaced from its equilibrium position by a vertical distance and then released to vibrate freely.



(a) If $y_i(t)$ denotes the displacement of m_i from equilibrium at time t, show that the mass-stiffness equation is $\mathbf{My}'' = \mathbf{Ky}$, where

$$\mathbf{M}=\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \ \mathbf{y}=\begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}, \ \mathbf{K}=k\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

($k_{33} = 1$ is not a mistake!).

(b) Show that $\mathbf{K}$ is positive definite.

(c) Find the normal modes when $m_1 = m_2 = m_3 = m$.

7.6.4. By diagonalizing the quadratic form $13x^2 + 10xy + 13y^2$, show that the rotated graph of $13x^2 + 10xy + 13y^2 = 72$ is an ellipse in standard form as shown in Figure 7.2.1 on p. 505.

7.6.5. Suppose that $\mathbf{A}$ is a real-symmetric matrix. Explain why the signs of the pivots in the LDU factorization for $\mathbf{A}$ reveal the inertia of $\mathbf{A}$.