the form $\mathbf{A} = r\mathbf{I} - \mathbf{B}$, where $\mathbf{B} \geq \mathbf{0}$ and $r > \rho(\mathbf{B})$, then (7.10.14) guarantees that $\mathbf{A}^{-1}$ exists and $\mathbf{A}^{-1} \geq \mathbf{0}$, and it's clear that $a_{ij} \leq 0$ for each $i \neq j$, so $\mathbf{A}$ must be an M-matrix. ∎

Proof of (7.10.26). If $\mathbf{A}$ is an M-matrix, then, by (7.10.25), $\mathbf{A} = r\mathbf{I} - \mathbf{B}$, where $r > \rho(\mathbf{B})$. This means that if $\lambda_{\mathbf{A}} \in \sigma(\mathbf{A})$, then $\lambda_{\mathbf{A}} = r - \lambda_{\mathbf{B}}$ for some $\lambda_{\mathbf{B}} \in \sigma(\mathbf{B})$. If $\lambda_{\mathbf{B}} = \alpha + \mathrm{i}\beta$, then $r > \rho(\mathbf{B}) \geq |\lambda_{\mathbf{B}}| = \sqrt{\alpha^2 + \beta^2} \geq |\alpha| \geq \alpha$ implies that $\operatorname{Re}(\lambda_{\mathbf{A}}) = r - \alpha \geq 0$. Now suppose that $\mathbf{A}$ is any matrix such that $a_{ij} \leq 0$ for all $i \neq j$ and $\operatorname{Re}(\lambda_{\mathbf{A}}) > 0$ for all $\lambda_{\mathbf{A}} \in \sigma(\mathbf{A})$. This means that there is a real number γ such that the circle centered at γ and having radius equal to γ contains $\sigma(\mathbf{A})$—see Figure 7.10.1. Let r be any real number such that $r > \max\{2\gamma, \max_i |a_{ii}|\}$, and set $\mathbf{B} = r\mathbf{I} - \mathbf{A}$. It's apparent that $\mathbf{B} \geq \mathbf{0}$, and, as can be seen from Figure 7.10.1, the distance $|r - \lambda_{\mathbf{A}}|$ between r and every point in $\sigma(\mathbf{A})$ is less than r.

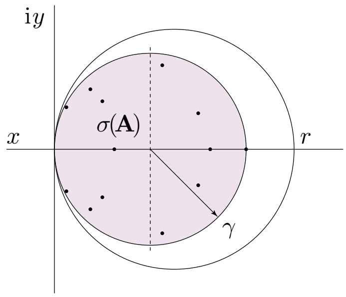

FIGURE 7.10.1

All eigenvalues of $\mathbf{B}$ look like $\lambda_{\mathbf{B}} = r - \lambda_{\mathbf{A}}$, and $|\lambda_{\mathbf{B}}| = |r - \lambda_{\mathbf{A}}| < r$, so $\rho(\mathbf{B}) < r$. Since $\mathbf{A} = r\mathbf{I} - \mathbf{B}$ is nonsingular (because $0 \notin \sigma(\mathbf{A})$) with $\mathbf{B} \geq \mathbf{0}$ and $r > \rho(\mathbf{B})$, it follows from (7.10.14) in Example 7.10.3 (p. 620) that $\mathbf{A}^{-1} \geq \mathbf{0}$, and thus $\mathbf{A}$ is an M-matrix. ∎

Proof of (7.10.27). If $\widetilde{\mathbf{A}}_{k\times k}$ is the principal submatrix lying on the intersection of rows and columns $i_1, \ldots, i_k$ in an M-matrix $\mathbf{A} = r\mathbf{I} - \mathbf{B}$, where $\mathbf{B} \geq \mathbf{0}$ and $r > \rho(\mathbf{B})$, then $\widetilde{\mathbf{A}} = r\mathbf{I} - \widetilde{\mathbf{B}}$, where $\widetilde{\mathbf{B}} \geq \mathbf{0}$ is the corresponding principal submatrix of $\mathbf{B}$. Let $\mathbf{P}$ be a permutation matrix such that

$$\mathbf{P}^T\mathbf{B}\mathbf{P} = \begin{pmatrix} \widetilde{\mathbf{B}} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix}, \text{ or } \mathbf{B} = \mathbf{P}\begin{pmatrix} \widetilde{\mathbf{B}} & \mathbf{X} \\ \mathbf{Y} & \mathbf{Z} \end{pmatrix}\mathbf{P}^T, \text{ and let } \mathbf{C} = \mathbf{P}\begin{pmatrix} \widetilde{\mathbf{B}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{P}^T.$$

Clearly, $\mathbf{0} \leq \mathbf{C} \leq \mathbf{B}$, so, by (7.10.13) on p. 619, $\rho(\widetilde{\mathbf{B}}) = \rho(\mathbf{C}) \leq \rho(\mathbf{B}) < r$. Consequently, (7.10.25) insures that $\widetilde{\mathbf{A}}$ is an M-matrix. ∎

Proof of (7.10.28). If $\mathbf{A}$ is an M-matrix, then $\det(\mathbf{A}) > 0$ because the eigenvalues of a real matrix appear in complex conjugate pairs, so (7.10.26) and (7.1.8),