To analyze limiting properties of Markov chains, divide the class of stochastic matrices (and hence the class of stationary Markov chains) into four mutually exclusive categories as described below.

(1)	Irreducible	with	$\lim_{k\to\infty} \mathbf{P}^k$ existing	(i.e., $\mathbf{P}$ is primitive).
(2)	Irreducible	with	$\lim_{k\to\infty} \mathbf{P}^k$ not existing	(i.e., $\mathbf{P}$ is imprimitive).
(3)	Reducible	with	$\lim_{k\to\infty} \mathbf{P}^k$ existing.	
(4)	Reducible	with	$\lim_{k\to\infty} \mathbf{P}^k$ not existing.	

In case (1), where $\mathbf{P}$ is primitive, we know exactly what $\lim_{k\to\infty} \mathbf{P}^k$ looks like. The Perron vector for $\mathbf{P}$ is $\mathbf{e}/n$ (the uniform distribution vector), so if $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)^T$ is the Perron vector for $\mathbf{P}^T$, then

$$\lim_{k\to\infty} \mathbf{P}^k = \frac{(\mathbf{e}/n)\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T(\mathbf{e}/n)} = \frac{\mathbf{e}\boldsymbol{\pi}^T}{\boldsymbol{\pi}^T\mathbf{e}} = \mathbf{e}\boldsymbol{\pi}^T = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \\ \pi_1 & \pi_2 & \cdots & \pi_n \\ \vdots & \vdots & & \vdots \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix} > \mathbf{0} \tag{8.4.3}$$

by (8.3.10) on p. 674. Therefore, if $\mathbf{P}$ is primitive, then a limiting probability distribution exists, and it is given by

$$\lim_{k\to\infty} \mathbf{p}^T(k) = \lim_{k\to\infty} \mathbf{p}^T(0)\mathbf{P}^k = \mathbf{p}^T(0)\mathbf{e}\boldsymbol{\pi}^T = \boldsymbol{\pi}^T. \tag{8.4.4}$$

Notice that because $\sum_k p_k(0) = 1$, the term $\mathbf{p}^T(0)\mathbf{e}$ drops away, so we have the conclusion that the value of the limit is *independent* of the value of the initial distribution $\mathbf{p}^T(0)$, which isn't too surprising.

Example 8.4.2

Going back to the mouse-in-the-box example, it's easy to confirm that the transition matrix $\mathbf{M}$ in (8.4.1) is primitive, so $\lim_{k\to\infty} \mathbf{M}^k$ as well as $\lim_{k\to\infty} \mathbf{p}^T(0)$ must exist, and their values are determined by the left-hand Perron vector of $\mathbf{M}$ that can be found by calculating any nonzero vector $\mathbf{v} \in N\left(\mathbf{I} - \mathbf{M}^T\right)$ and normalizing it to produce $\boldsymbol{\pi}^T = \mathbf{v}^T/\left\|\mathbf{v}\right\|_1$. Routine computation reveals that the one solution of the homogeneous equation $(\mathbf{I} - \mathbf{M}^T)\mathbf{v} = \mathbf{0}$ is $\mathbf{v}^T = (2,\ 3,\ 3)$, so $\boldsymbol{\pi}^T = (1/8)(2,\ 3,\ 3)$, and thus

$$\lim_{k\to\infty} \mathbf{M}^k = \frac{1}{8}\begin{pmatrix} 2 & 3 & 3 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix} \quad \text{and} \quad \lim_{k\to\infty} \mathbf{p}^T(k) = \frac{1}{8}(2,\ 3,\ 3).$$

This limiting distribution can be interpreted as meaning that in the long run the mouse will occupy chamber #1 one-fourth of the time, while 37.5% of the time it's in chamber #2, and 37.5% of the time it's in chamber #3, and this is independent of where (or how) the process started. The mathematical justification for this statement is on p. 693.
