3.5 MATRIX MULTIPLICATION

The purpose of this section is to further develop the concept of matrix multiplication as introduced in the previous section. In order to do this, it is helpful to begin by composing a single row with a single column. If

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

the ***standard inner product*** of $\mathbf{R}$ with $\mathbf{C}$ is defined to be the scalar

$$\mathbf{RC} = r_1 c_1 + r_2 c_2 + \cdots + r_n c_n = \sum_{i=1}^{n} r_i c_i.$$

For example,

$$\begin{pmatrix} 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2)(1) + (4)(2) + (-2)(3) = 4.$$

Recall from (3.4.1) that the product of two 2×2 matrices

$$\mathbf{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{G} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

was defined naturally by writing

$$\mathbf{FG} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix} = \mathbf{H}.$$

Notice that the (i, j)-entry in the product $\mathbf{H}$ can be described as the inner product of the i^{th} row of $\mathbf{F}$ with the j^{th} column in $\mathbf{G}$. That is,

$$h_{11} = \mathbf{F}_{1*}\mathbf{G}_{*1} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}, \qquad h_{12} = \mathbf{F}_{1*}\mathbf{G}_{*2} = \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix},$$

$$h_{21} = \mathbf{F}_{2*}\mathbf{G}_{*1} = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix}, \qquad h_{22} = \mathbf{F}_{2*}\mathbf{G}_{*2} = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} B \\ D \end{pmatrix}.$$

This is exactly the way that the general definition of matrix multiplication is formulated.