8.5 PERRON COMPLEMENTATION

The purpose of this section is to show how the Perron vector of an irreducible matrix $\mathbf{A} \geq \mathbf{0}$ may be obtained by gluing together Perron vectors of smaller matrices called *Perron complements*.

Perron Complements

Partition an irreducible $\mathbf{A}_{n\times n} \geq \mathbf{0}$ with spectral radius $\rho(\mathbf{A}) = r$, as

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1k} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \mathbf{A}_{k2} & \cdots & \mathbf{A}_{kk} \end{pmatrix}, \tag{8.5.1}$$

where all diagonal blocks are square. The ***Perron complement*** of the i^{th} diagonal block $\mathbf{A}_{ii}$ is defined to be the matrix

$$\mathbf{P}_i = \mathbf{A}_{ii} + \mathbf{A}_{i\star}(r\mathbf{I} - \mathbf{A}_i^{\star})^{-1}\mathbf{A}_{\star i}, \tag{8.5.2}$$

where $\mathbf{A}_{i\star}$ and $\mathbf{A}_{\star i}$ are, respectively, the i^{th} row and the i^{th} column of blocks with $\mathbf{A}_{ii}$ removed, and $\mathbf{A}_i^{\star}$ is the principal submatrix of $\mathbf{A}$ obtained by deleting the i^{th} row and i^{th} column of blocks. The nonsingularity of $r\mathbf{I} - \mathbf{A}_i^{\star}$ is established in (8.5.4) on p. 707.

Example 8.5.1

If an irreducible $\mathbf{A} \geq \mathbf{0}$ with $\rho(\mathbf{A}) = r$ is partitioned as $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$, then the two Perron complements are

$$\mathbf{P}_1 = \mathbf{A}_{11} + \mathbf{A}_{12}(r\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21} \quad \text{and} \quad \mathbf{P}_2 = \mathbf{A}_{22} + \mathbf{A}_{21}(r\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{A}_{12}.$$

If $\mathbf{A}$ is partitioned as $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{23} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix}$, then there are three Perron complements, and the second one is

$$\mathbf{P}_2 = \mathbf{A}_{22} + \begin{pmatrix} \mathbf{A}_{21} & \mathbf{A}_{23} \end{pmatrix} \begin{pmatrix} r\mathbf{I} - \mathbf{A}_{11} & -\mathbf{A}_{13} \\ -\mathbf{A}_{31} & r\mathbf{I} - \mathbf{A}_{33} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{A}_{12} \\ \mathbf{A}_{32} \end{pmatrix}.$$

The other complements, $\mathbf{P}_1$ and $\mathbf{P}_3$, are similarly written. Although Perron complements are not the same as the Schur complements defined on page 123, there is a connection. For example, if $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$, then

$$\mathbf{P}_1 = r\mathbf{I} - \text{the Schur complement of } (r\mathbf{I} - \mathbf{A}_{22}) \quad \text{in the matrix } r\mathbf{I} - \mathbf{A},$$

and similarly for $\mathbf{P}_2$. This is not the only reason for the terminology *Perron complement*—the other reasons will become evident as later developments unfold.

The salient feature of all Perron complements is that they inherit the same "Perron properties" from their parent matrix. In other words, if $\mathbf{A}$ is nonnegative and irreducible, then so is each Perron complement $\mathbf{P}_i$ that's derived from $\mathbf{A}$. Furthermore, if $\rho(\mathbf{A}) = r$, then $\rho(\mathbf{P}_i) = r$ for each i. And, most importantly, the Perron vectors of the $\mathbf{P}_i$'s combine to form the Perron vector of the parent matrix $\mathbf{A}$.

Before we can understand why these things are true, a few preliminary results are needed, some of which are of interest in their own right. The first such result is the converse to part of the Perron–Frobenius Theorem on p. 673.

Irreducibility

$\mathbf{A}_{n\times n} \geq \mathbf{0}$ is irreducible if and only if $\mathbf{A}$ has a simple positive eigenvalue $\lambda > 0$ that is associated with a positive right-hand eigenvector $\mathbf{p} > \mathbf{0}$ as well as a positive left-hand eigenvector $\mathbf{q}^T > \mathbf{0}$.

Proof. If $\mathbf{p} > \mathbf{0}$ and $\mathbf{q}^T > \mathbf{0}$ are respective right-hand and left-hand eigenvectors for $\mathbf{A}$ that are associated with the simple eigenvalue $\lambda > 0$, and if $\mathbf{D} = \operatorname{diag}(p_1, p_2, \ldots, p_n)$ has the components of $\mathbf{p}$ as diagonal entries, then

$$\mathbf{P} = \frac{\mathbf{D}^{-1}\mathbf{A}\mathbf{D}}{\lambda}$$

is a stochastic matrix, and 1 is a simple eigenvalue of $\mathbf{P}$ associated with the respective right-hand and left-hand eigenvectors $\mathbf{D}^{-1}\mathbf{p} = \mathbf{e} > \mathbf{0}$ and $\mathbf{q}^T\mathbf{D} > \mathbf{0}$ (as in earlier sections, $\mathbf{e}$ is a column of 1's). Clearly, $\mathbf{P}$ is irreducible if and only if $\mathbf{A}$ is irreducible. All stochastic matrices are Cesàro summable to the spectral projector $\mathbf{G}$ onto $N(\mathbf{I}-\mathbf{P})$ along $R(\mathbf{I}-\mathbf{P})$ (p. 697), and the simplicity of $1 \in \sigma(\mathbf{P})$ implies that

$$\mathbf{G} = \frac{\mathbf{D}^{-1}\mathbf{p}\mathbf{q}^T\mathbf{D}}{\mathbf{q}^T\mathbf{p}} > \mathbf{0} \qquad \text{(recall (7.2.12) on p. 518).}$$

It's therefore impossible for $\mathbf{P}$ (and hence $\mathbf{A}$) to be reducible because otherwise there is a position (i,j) with $i \neq j$ such that for all $k = 1, 2, \ldots$

$$\left[\mathbf{P}^k\right]_{ij} = 0 \implies \left[\frac{\mathbf{I}+\mathbf{P}+\cdots+\mathbf{P}^{k-1}}{k}\right]_{ij} = 0 \text{ for } k = 1, 2, \ldots \implies \mathbf{G}_{ij} = 0.$$

Thus sufficient conditions for irreducibility are established—necessity of these conditions follows from the Perron–Frobenius theorem on p. 673. ∎

In order for the Perron complement $\mathbf{P}_i = \mathbf{A}_{ii} + \mathbf{A}_{i\star}(r\mathbf{I} - \mathbf{A}_i^\star)^{-1}\mathbf{A}_{\star i}$ to be well defined, the existence of $(r\mathbf{I} - \mathbf{A}_i^\star)^{-1}$ must be ensured. This, along with the fact that $(r\mathbf{I} - \mathbf{A}_i^\star)^{-1} > \mathbf{0}$, is the point of the following theorem.

Principal Submatrices

If $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is an irreducible matrix with $\rho(\mathbf{A}) = r$ that is partitioned as in (8.5.1), and if $\mathbf{A}_i^\star$ is the principal submatrix of $\mathbf{A}$ obtained by deleting the i^{th} row and i^{th} column of blocks, then

- $\rho(\mathbf{A}_i^\star) < r,$ (8.5.3)
- $(r\mathbf{I} - \mathbf{A}_i^\star)$ is nonsingular, and $(r\mathbf{I} - \mathbf{A}_i^\star)^{-1} > \mathbf{0}.$ (8.5.4)

In other words, $r\mathbf{I} - \mathbf{A}_i^\star$ is an M-matrix—see Example 7.10.7, p. 626.

Proof. To prove that $\rho(\mathbf{A}_i^\star) < r$, suppose to the contrary that $r \leq \rho(\mathbf{A}_i^\star)$. If $\mathbf{Q}$ is the permutation matrix such that

$$\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \begin{pmatrix} \mathbf{A}_i^\star & \mathbf{A}_{\star i} \\ \mathbf{A}_{i\star} & \mathbf{A}_{ii} \end{pmatrix} = \widetilde{\mathbf{A}}, \quad \text{and if} \quad \widetilde{\mathbf{B}} = \begin{pmatrix} \mathbf{A}_i^\star & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{8.5.5}$$

then $\rho(\widetilde{\mathbf{A}}) = \rho(\mathbf{A}) = r \leq \rho(\mathbf{A}_i^\star) = \rho(\widetilde{\mathbf{B}})$. Furthermore $\widetilde{\mathbf{A}} \geq \widetilde{\mathbf{B}} \geq \mathbf{0}$ ensures that $\rho(\widetilde{\mathbf{A}}) \geq \rho(\widetilde{\mathbf{B}})$ (Example 7.10.2, p. 619), so $r = \rho(\widetilde{\mathbf{B}}) = \rho(\mathbf{A}_i^\star)$. But this impossible because (8.3.2) guarantees the existence of a vector $\mathbf{v} \geq \mathbf{0}$, $\mathbf{v} \neq \mathbf{0}$, such that $\mathbf{A}_i^\star\mathbf{v} = r\mathbf{v}$, so $\mathbf{z} = (\mathbf{v} \;\; \mathbf{0})^T$ is a nonnegative nonzero vector $\mathbf{v}$ such that $\widetilde{\mathbf{B}}\mathbf{z} = r\mathbf{z}$, and

$$\widetilde{\mathbf{A}} \geq \widetilde{\mathbf{B}} \Longrightarrow \widetilde{\mathbf{A}}\mathbf{z} \geq \widetilde{\mathbf{B}}\mathbf{z} = r\mathbf{z} \Longrightarrow \widetilde{\mathbf{A}}\mathbf{z} = r\mathbf{z} \quad \text{and } \mathbf{z} > \mathbf{0} \quad \text{(Example 8.3.1, p. 674).}$$

Thus $\rho(\mathbf{A}_i^\star) < r$. The fact that $(r\mathbf{I} - \mathbf{A}_i^\star)$ is nonsingular and $(r\mathbf{I} - \mathbf{A}_i^\star)^{-1} > \mathbf{0}$ follows from (7.10.14) on p. 620. ∎

Note: $\rho(\mathbf{A}_i^\star) < r$ is also a consequence of Wielandt's Theorem (p. 675).

Some of the properties that Perron complements inherit from their parent matrix are now easily established.

Inherited Perron Properties

If $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is an irreducible matrix with $\rho(\mathbf{A}) = r$ that is partitioned as in (8.5.1), and if $\mathbf{P}_i = \mathbf{A}_{ii} + \mathbf{A}_{i\star}(r\mathbf{I} - \mathbf{A}_i^\star)^{-1}\mathbf{A}_{\star i}$ is the i^{th} Perron complement as defined in (8.5.2), then

- $\mathbf{P}_i \geq \mathbf{0}$ for every i, (8.5.6)
- $\mathbf{P}_i$ is irreducible for every i, (8.5.7)
- $\rho(\mathbf{P}_i) = r$ for every i. (8.5.8)

Proof. $\mathbf{P}_i \geq \mathbf{0}$ because $(r\mathbf{I} - \mathbf{A}_i^\star)^{-1} > \mathbf{0}$, and all of the other terms in $\mathbf{P}_i$ are nonnegative. To see that $\mathbf{P}_i$ is irreducible, let $\widetilde{\mathbf{p}} = \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ be the partitioned right-hand Perron vector for the nonnegative irreducible matrix $\widetilde{\mathbf{A}}$ given in (8.5.5) so that $(r\mathbf{I} - \widetilde{\mathbf{A}})\widetilde{\mathbf{p}} = \mathbf{0}$. Consequently, the lower part of

$$\begin{pmatrix} r\mathbf{I} - \mathbf{A}_i^\star & -\mathbf{A}_{\star i} \\ -\mathbf{A}_{i\star} & r\mathbf{I} - \mathbf{A}_{ii} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \Longrightarrow \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_{i\star}(r\mathbf{I} - \mathbf{A}_i^\star)^{-1} & \mathbf{I} \end{pmatrix}\begin{pmatrix} r\mathbf{I} - \mathbf{A}_i^\star & -\mathbf{A}_{\star i} \\ -\mathbf{A}_{i\star} & r\mathbf{I} - \mathbf{A}_{ii} \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}$$

yields

$$(r\mathbf{I} - \mathbf{P}_i)\mathbf{y} = \mathbf{0}, \tag{8.5.9}$$

and thus $(r, \mathbf{y})$ is a right-hand eigenpair for $\mathbf{P}_i$ with $\mathbf{y} > \mathbf{0}$. A similar argument shows that there is also a left-hand eigenpair $(r, \mathbf{z}^T)$ for $\mathbf{P}_i$ with $\mathbf{z}^T > \mathbf{0}$. Furthermore, r is a simple eigenvalue of $\mathbf{P}_i$ because Perron–Frobenius insures that r is a simple eigenvalue of $\mathbf{A}$, as well as $\widetilde{\mathbf{A}}$, so this together with

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_{i\star}(r\mathbf{I} - \mathbf{A}_i^\star)^{-1} & \mathbf{I} \end{pmatrix}\begin{pmatrix} r\mathbf{I} - \mathbf{A}_i^\star & -\mathbf{A}_{\star i} \\ -\mathbf{A}_{i\star} & r\mathbf{I} - \mathbf{A}_{ii} \end{pmatrix}\begin{pmatrix} \mathbf{I} & (r\mathbf{I} - \mathbf{A}_i^\star)^{-1}\mathbf{A}_{\star i} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} = \begin{pmatrix} r\mathbf{I} - \mathbf{A}_i^\star & \mathbf{0} \\ \mathbf{0} & r\mathbf{I} - \mathbf{P}_i \end{pmatrix}$$

and the fact that $(r\mathbf{I} - \mathbf{A}_i^\star)$ is nonsingular produces

$$1 = \dim N(r\mathbf{I} - \widetilde{\mathbf{A}}) = \dim N(r\mathbf{I} - \mathbf{A}_i^\star) + \dim N(r\mathbf{I} - \mathbf{P}_i) = \dim N(r\mathbf{I} - \mathbf{P}_i).$$

The irreducibility of $\mathbf{P}_i$ now follows from the result on page 706. Finally, part of the Perron–Frobenius theorem (p. 673) states that a nonnegative irreducible matrix can have no nonnegative eigenvectors other than multiples of the positive Perron vector associated with the spectral radius. Therefore, since $(r, \mathbf{y})$ is an eigenpair for $\mathbf{P}_i$ with $\mathbf{y} > \mathbf{0}$, it follows that $\rho(\mathbf{P}_i) = r$, where $\mathbf{z}_i = \mathbf{y}/\|\mathbf{y}\|_1$ is the associated Perron vector. ∎

The above proof is more important than it might first appear to be because it reveals a significant relationship between the Perron vector of $\mathbf{A}$ and the Perron vector of $\mathbf{P}_i$. If the Perron vector for $\mathbf{A}$ is partitioned conformably with the partition in (8.5.1) as

$$\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_k \end{pmatrix},$$

then the nature of the permutation in (8.5.5) makes it's clear that $\mathbf{p}_i = \mathbf{y}$, where $\mathbf{y} > \mathbf{0}$ is the vector in (8.5.9). Consequently, the Perron vector for $\mathbf{P}_i$ is $\mathbf{z}_i = \mathbf{y}/\|\mathbf{y}\|_1 = \mathbf{y}/\mathbf{e}^T\mathbf{y} = \mathbf{p}_i/\mathbf{e}^T\mathbf{p}_i$ ($\mathbf{e}$ is a vector of 1's), or, equivalently,

$$\mathbf{p}_i = \xi_i \mathbf{z}_i, \quad \text{where} \quad \xi_i = \mathbf{e}^T\mathbf{p}_i. \tag{8.5.10}$$

In other words, the Perron vectors $\mathbf{z}_i$ of smaller Perron complements can be glued together with to build the Perron vector of $\mathbf{A}$ by writing

$$\mathbf{p} = \begin{pmatrix} \xi_1\mathbf{z}_1 \\ \xi_2\mathbf{z}_2 \\ \vdots \\ \xi_k\mathbf{z}_k \end{pmatrix}. \tag{8.5.11}$$

This looks like a nice result until you realize that the glue is the set of scalars $\xi_i = \mathbf{e}^T\mathbf{p}_i$, so we are going in circles if we need to use the components of $\mathbf{p}$ in order to compute the components $\mathbf{p}$. Fortunately, there's a clever way out of this dilemma by manufacturing the glue from the Perron vector of a *coupling matrix* $\mathbf{C}$, which is yet another matrix that inherits its Perron properties from the parent matrix $\mathbf{A}$. The following theorem brings everything together.

The Coupling Theorem

Suppose $\mathbf{A}_{n\times n} \geq \mathbf{0}$ is irreducible with $\rho(\mathbf{A}) = r$ that is partitioned into k levels as in (8.5.1). Let $\mathbf{p}$ and $\mathbf{z}_i$ be the respective Perron vectors of $\mathbf{A}$ and the Perron complement $\mathbf{P}_i$ defined in (8.5.2), and let $\mathbf{e}$ denote a column of 1's whose size is defined by the context. The matrix

$$\mathbf{C} = \begin{pmatrix} \mathbf{e}^T\mathbf{A}_{11}\mathbf{z}_1 & \cdots & \mathbf{e}^T\mathbf{A}_{1k}\mathbf{z}_k \\ \vdots & \ddots & \vdots \\ \mathbf{e}^T\mathbf{A}_{k1}\mathbf{z}_1 & \cdots & \mathbf{e}^T\mathbf{A}_{kk}\mathbf{z}_k \end{pmatrix}_{k\times k}$$

is called the ***coupling matrix***, and it has the following properties.

- $\mathbf{C}$ is nonnegative and irreducible.
- $\rho(\mathbf{C}) = r$.
- The Perron vector for $\mathbf{C}$, called the ***coupling vector,*** is given by

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_k \end{pmatrix}, \quad \text{where } \xi_i = \mathbf{e}^T\mathbf{p}_i \text{ is as defined in (8.5.10).}$$

- The Perron vector for $\mathbf{A}$ is given by $\mathbf{p} = \begin{pmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \vdots \\ \mathbf{p}_k \end{pmatrix} = \begin{pmatrix} \xi_1\mathbf{z}_1 \\ \xi_2\mathbf{z}_2 \\ \vdots \\ \xi_k\mathbf{z}_k \end{pmatrix}.$

Proof. Clearly, $\mathbf{C} \geq \mathbf{0}$ because each term $c_{ij} = \mathbf{e}^T\mathbf{A}_{ij}\mathbf{z}_j$ is nonnegative. Since $c_{ij} = 0 \iff \mathbf{A}_{ij} = \mathbf{0}$, it follows that $\mathbf{C}$ must be irreducible (if $\mathbf{C}$ could be permuted to a block triangular form, then so could $\mathbf{A}$). To prove the rest of the theorem, notice that $\mathbf{C} = \mathbf{RAL}$, where $\mathbf{R}$ and $\mathbf{L}$ are given by

$$\mathbf{R} = \begin{pmatrix} \mathbf{e}^T & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{e}^T & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{e}^T \end{pmatrix}_{k\times n} \quad \text{and} \quad \mathbf{L} = \begin{pmatrix} \mathbf{z}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{z}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{z}_k \end{pmatrix}_{n\times k}.$$

We know from (8.5.10) that $\mathbf{L}\boldsymbol{\xi} = \mathbf{p}$ and $\mathbf{Rp} = \boldsymbol{\xi}$, so

$$\mathbf{C}\boldsymbol{\xi} = \mathbf{RAL}\boldsymbol{\xi} = \mathbf{RAp} = \mathbf{R}(r\mathbf{p}) = r\boldsymbol{\xi}.$$

Furthermore, $\boldsymbol{\xi} > \mathbf{0}$ (because $\mathbf{p}_i > \mathbf{0}$ for each i), and $\mathbf{e}^T\boldsymbol{\xi} = \mathbf{e}^T\mathbf{Rp} = \mathbf{e}^T\mathbf{p} = 1$. It now follows that $r = \rho(\mathbf{C})$ and $\boldsymbol{\xi}$ is the Perron vector for $\mathbf{C}$. The conclusion that $\mathbf{p} = \begin{pmatrix} \xi_1\mathbf{z}_1 \\ \xi_2\mathbf{z}_2 \\ \vdots \\ \xi_k\mathbf{z}_k \end{pmatrix}$ comes from (8.5.11). ∎

Note: The matrices $\mathbf{R}$ and $\mathbf{L}$ are special cases of transformations known respectively as ***restriction*** and ***prolongation*** operations because when $n > k$, $\mathbf{R}$ "restricts" n-tuples down to k-tuples while $\mathbf{L}$ "prolongates" k-tuples back up to n-tuples in an inverse-like manner since $\mathbf{RL} = \mathbf{I}$. Restriction-prolongation techniques like the one above are popular tools in applied and numerical work.

Example 8.5.2

The matrix

$$\mathbf{A} = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 3 \\ 4 & 2 & 3 & 0 \\ \hline 0 & 3 & 2 & 4 \\ 3 & 0 & 1 & 2 \end{array}\right) = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix}$$

is irreducible with $\rho(\mathbf{A}) = 7$, and the two Perron complements are

$$\mathbf{P}_1 = \mathbf{A}_{11} + \mathbf{A}_{12}(7\mathbf{I} - \mathbf{A}_{22})^{-1}\mathbf{A}_{21} = \frac{1}{7}\begin{pmatrix} 29 & 10 \\ 40 & 29 \end{pmatrix}, \quad \text{with } \rho(\mathbf{P}_1) = 7,$$

and

$$\mathbf{P}_2 = \mathbf{A}_{22} + \mathbf{A}_{21}(7\mathbf{I} - \mathbf{A}_{11})^{-1}\mathbf{A}_{12} = \frac{1}{7}\begin{pmatrix} 29 & 40 \\ 10 & 29 \end{pmatrix}, \quad \text{with } \rho(\mathbf{P}_2) = 7.$$

The respective Perron vectors for $\mathbf{P}_1$ and $\mathbf{P}_2$ are

$$\mathbf{z}_1 = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \quad \text{and} \quad \mathbf{z}_2 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix},$$

and the coupling matrix is

$$\mathbf{C} = \begin{pmatrix} \mathbf{e}^T\mathbf{A}_{11}\mathbf{z}_1 & \mathbf{e}^T\mathbf{A}_{12}\mathbf{z}_2 \\ \mathbf{e}^T\mathbf{A}_{21}\mathbf{z}_1 & \mathbf{e}^T\mathbf{A}_{22}\mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 4 \end{pmatrix}, \quad \text{with } \rho(\mathbf{C}) = 7.$$

The coupling vector (the Perron vector of $\mathbf{C}$) is

$$\boldsymbol{\xi} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}, \quad \text{so the Perron vector of } \mathbf{A} \text{ is } \quad \mathbf{p} = \begin{pmatrix} \mathbf{z}_1/2 \\ \mathbf{z}_2/2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

Knowledge of $\rho(\mathbf{A})$ is required to form the Perron complements of $\mathbf{A}$, and this can be a bottleneck in some situations. However, there are important applications in which the spectral radius is known in advance. A notable example is the theory of finite Markov chains as described in §8.4 (p. 687) because $\rho(\mathbf{P}) = 1$ for all transition probability matrices $\mathbf{P}$. The next section is devoted to showing how Perron complementation is applied in the theory of Markov chains.

Exercises for section 8.5

8.5.1. Consider a nonnegative irreducible partitioned matrix and its associated coupling matrix

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1k} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{k1} & \cdots & \mathbf{A}_{kk} \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} \mathbf{e}^T\mathbf{A}_{11}\mathbf{z}_1 & \cdots & \mathbf{e}^T\mathbf{A}_{1k}\mathbf{z}_k \\ \vdots & \ddots & \vdots \\ \mathbf{e}^T\mathbf{A}_{k1}\mathbf{z}_1 & \cdots & \mathbf{e}^T\mathbf{A}_{kk}\mathbf{z}_k \end{pmatrix}.$$

Prove that $\mathbf{A}$ is primitive if and only if $\mathbf{C}$ is primitive.

8.5.2. Show the coupling properties hold for more general matrices with eigenvalue λ
—Take $\mathbf{R} = \mathbf{L}^T$ in $\mathbf{C} = \mathbf{RAL}$, and use 2-norms

8.5.3. Let $\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{c} \\ \mathbf{d}^T & \alpha \end{pmatrix}$ with $\mathbf{c}, \mathbf{d} \neq \mathbf{0}$, and suppose that $r \in \sigma(\mathbf{A})$ but $r \notin \sigma(\mathbf{A}_1)$. Show that $\mathbf{z}^T = \left((r\mathbf{I} - \mathbf{A}_1)^{-1}\mathbf{c} \,|\, 1\right)$ is an eigenvector for $\mathbf{A}$ associated with r.

8.5.4. Give an alternate argument for the irreducibility of Perron complements using graph theory.