Solutions for Chapter 8

Solutions for exercises in section 8. 2

8.2.1. The eigenvalues are $\sigma(\mathbf{A}) = \{12, 6\}$ with $alg\ mult_{\mathbf{A}}(6) = 2$, and it's clear that $12 = \rho(\mathbf{A}) \in \sigma(\mathbf{A})$. The eigenspace $N(\mathbf{A} - 12\mathbf{I})$ is spanned by $\mathbf{e} = (1, 1, 1)^T$, so the Perron vector is $\mathbf{p} = (1/3)(1, 1, 1)^T$. The left-hand eigenspace $N(\mathbf{A}^T - 12\mathbf{I})$ is spanned by $(1, 2, 3)^T$, so the left-hand Perron vector is $\mathbf{q}^T = (1/6)(1, 2, 3)$.

8.2.3. If $\mathbf{p}_1$ and $\mathbf{p}_2$ are two vectors satisfying $\mathbf{Ap} = \rho(\mathbf{A})\mathbf{p}$, $\mathbf{p} > \mathbf{0}$, and $\|\mathbf{p}\|_1 = 1$, then $\dim N(\mathbf{A} - \rho(\mathbf{A})\mathbf{I}) = 1$ implies that $\mathbf{p}_1 = \alpha\mathbf{p}_2$ for some $\alpha < 0$. But $\|\mathbf{p}_1\|_1 = \|\mathbf{p}_2\|_1 = 1$ insures that $\alpha = 1$.

8.2.4. $\sigma(\mathbf{A}) = \{0, 1\}$, so $\rho(\mathbf{A}) = 1$ is the Perron root, and the Perron vector is $\mathbf{p} = (\alpha + \beta)^{-1}(\beta, \alpha)$.

8.2.5. (a) $\rho(\mathbf{A}/r) = 1$ is a simple eigenvalue of $\mathbf{A}/r$, and it's the only eigenvalue on the spectral circle of $\mathbf{A}/r$, so (7.10.33) on p. 630 guarantees that $\lim_{k\to\infty}(\mathbf{A}/r)^k$ exists.

(b) This follows from (7.10.34) on p. 630.

(c) $\mathbf{G}$ is the spectral projector associated with the simple eigenvalue $\lambda = r$, so formula (7.2.12) on p. 518 applies.

8.2.6. If $\mathbf{e}$ is the column of all 1 's, then $\mathbf{Ae} = \rho\mathbf{e}$. Since $\mathbf{e} > \mathbf{0}$, it must be a positive multiple of the Perron vector $\mathbf{p}$, and hence $\mathbf{p} = n^{-1}\mathbf{e}$. Therefore, $\mathbf{Ap} = \rho\mathbf{p}$ implies that $\rho = \rho(\mathbf{A})$. The result for column sums follows by considering $\mathbf{A}^T$.

8.2.7. Since $\rho = \max_i \sum_j a_{ij}$ is the largest row sum of $\mathbf{A}$, there must exist a matrix $\mathbf{E} \geq \mathbf{0}$ such that every row sum of $\mathbf{B} = \mathbf{A} + \mathbf{E}$ is ρ. Use Example 7.10.2 (p. 619) together with Exercise 8.2.7 to obtain $\rho(\mathbf{A}) \leq \rho(\mathbf{B}) = \rho$. The lower bound follows from the Collatz–Wielandt formula. If $\mathbf{e}$ is the column of ones, then $\mathbf{e} \in \mathcal{N}$, so

$$\rho(\mathbf{A}) = \max_{\mathbf{x}\in\mathcal{N}} f(\mathbf{x}) \geq f(\mathbf{e}) = \min_{1\leq i\leq n} \frac{[\mathbf{Ae}]_i}{e_i} = \min_i \sum_{j=1}^{n} a_{ij}.$$

8.2.8. (a), (b), (c), and (d) are illustrated by using the nilpotent matrix $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(e) $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues ± 1.

8.2.9. If $\xi = g(\mathbf{x})$ for $\mathbf{x} \in \mathcal{P}$, then $\xi\mathbf{x} \geq \mathbf{Ax} > \mathbf{0}$. Let $\mathbf{p}$ and $\mathbf{q}^T$ be the respective the right-hand and left-hand Perron vectors for $\mathbf{A}$ associated with the Perron root r, and use (8.2.3) along with $\mathbf{q}^T\mathbf{x} > 0$ to write

$$\xi\mathbf{x} \geq \mathbf{Ax} > \mathbf{0} \implies \xi\mathbf{q}^T\mathbf{x} \geq \mathbf{q}^T\mathbf{Ax} = r\mathbf{q}^T\mathbf{x} \implies \xi \geq r,$$

so $g(\mathbf{x}) \geq r$ for all $\mathbf{x} \in \mathcal{P}$. Since $g(\mathbf{p}) = r$ and $\mathbf{p} \in \mathcal{P}$, it follows that $r = \min_{\mathbf{x}\in\mathcal{P}} g(\mathbf{x})$.

8.2.10. $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \implies \rho(\mathbf{A}) = 5,$ but $g(\mathbf{e}_1) = 1 \implies \min_{\mathbf{x}\in\mathcal{N}} g(\mathbf{x}) < \rho(\mathbf{A})$.

Solutions for exercises in section 8. 3

8.3.1. (a) The graph is strongly connected.

(b) $\rho(\mathbf{A}) = 3,$ and $\mathbf{p} = (1/6,\, 1/2,\, 1/3)^T$.

(c) $h = 2$ because $\mathbf{A}$ is imprimitive and singular.

8.3.2. If $\mathbf{A}$ is nonsingular then there are either one or two distinct nonzero eigenvalues inside the spectral circle. But this is impossible because $\sigma(\mathbf{A})$ has to be invariant under rotations of $120°$ by the result on p. 677. Similarly, if $\mathbf{A}$ is singular with $alg\ mult_{\mathbf{A}}(0) = 1,$ then there is a single nonzero eigenvalue inside the spectral circle, which is impossible.

8.3.3. No! The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has $\rho(\mathbf{A}) = 2$ with a corresponding eigenvector $\mathbf{e} = (1,1)^T$, but $\mathbf{A}$ is reducible.

8.3.4. $\mathbf{P}_n$ is nonnegative and irreducible (its graph is strongly connected), and $\mathbf{P}_n$ is imprimitive because $\mathbf{P}_n^n = \mathbf{I}$ insures that every power has zero entries. Furthermore, if $\lambda \in \sigma(\mathbf{P}_n)$, then $\lambda^n \in \sigma(\mathbf{P}_n^n) = \{1\}$, so all eigenvalues of $\mathbf{P}_n$ are roots of unity. Since all eigenvalues on the spectral circle are simple (recall (8.3.13) on p. 676) and uniformly distributed, it must be the case that $\sigma(\mathbf{P}_n) = \{1,\, \omega,\, \omega^2,\, \ldots,\, \omega^{n-1}\}$.

8.3.5. $\mathbf{A}$ is irreducible because the graph $\mathcal{G}(\mathbf{A})$ is strongly connected—every node is accessible by some sequence of paths from every other node.

8.3.6. $\mathbf{A}$ is imprimitive. This is easily seen by observing that each $\mathbf{A}^{2n}$ for $n > 1$ has the same zero pattern (and each $\mathbf{A}^{2n+1}$ for $n > 0$ has the same zero pattern), so every power of $\mathbf{A}$ has zero entries.

8.3.7. (a) Having row sums less than or equal to 1 means that $\|\mathbf{P}\|_\infty \leq 1$. Because $\rho(\star) \leq \|\star\|$ for every matrix norm (recall (7.1.12) on p. 497), it follows that $\rho(\mathbf{S}) \leq \|\mathbf{S}\|_1 \leq 1$.

(b) If $\mathbf{e}$ denotes the column of all 1's, then the hypothesis insures that $\mathbf{Se} \leq \mathbf{e}$, and $\mathbf{Se} \neq \mathbf{e}$. Since $\mathbf{S}$ is irreducible, the result in Example 8.3.1 (p. 674) implies that it's impossible to have $\rho(\mathbf{S}) = 1$ (otherwise $\mathbf{Se} = \mathbf{e}$), and therefore $\rho(\mathbf{S}) < 1$ by part (a).

8.3.8. If $\mathbf{p}$ is the Perron vector for $\mathbf{A}$, and if $\mathbf{e}$ is the column of 1's, then

$$\mathbf{D}^{-1}\mathbf{ADe} = \mathbf{D}^{-1}\mathbf{Ap} = r\mathbf{D}^{-1}\mathbf{p} = r\mathbf{e}$$

shows that every row sum of $\mathbf{D}^{-1}\mathbf{AD}$ is r, so we can take $\mathbf{P} = r^{-1}\mathbf{D}^{-1}\mathbf{AD}$ because the Perron–Frobenius theorem guarantees that $r > 0$.

8.3.9. Construct the Boolean matrices as described in Example 8.3.5 (p. 680), and show that $\mathbf{B}_9$ has a zero in the $(1,1)$ position, but $\mathbf{B}_{10} > \mathbf{0}$.

8.3.10. According to the discussion on p. 630, $\mathbf{f}(t) \to \mathbf{0}$ if $r < 1$. If $r = 1$, then $\mathbf{f}(t) \to \mathbf{Gf}(0) = \mathbf{p}\Big(\mathbf{q}^T\mathbf{f}(0)/\mathbf{q}^T\mathbf{p}\Big) > \mathbf{0}$, and if $r > 1$, the results of the Leslie analysis imply that $f_k(t) \to \infty$ for each k.

8.3.11. The only nonzero coefficient in the characteristic equation for $\mathbf{L}$ is c_1, so $\gcd\{2, 3, \ldots, n\} = 1$.

8.3.12. (a) Suppose that $\mathbf{A}$ is essentially positive. Since we can always find a $\beta > 0$ such that $\beta\mathbf{I} + \operatorname{diag}(a_{11}, a_{22}, \ldots, a_{nn}) \geq \mathbf{0}$, and since $a_{ij} \geq 0$ for $i \neq j$, it follows that $\mathbf{A} + \beta\mathbf{I}$ is a nonnegative irreducible matrix, so (8.3.5) on p. 672 can be applied to conclude that $(\mathbf{A} + (1+\beta)\mathbf{I})^{n-1} > \mathbf{0}$, and thus $\mathbf{A} + \alpha\mathbf{I}$ is primitive with $\alpha = \beta + 1$. Conversely, if $\mathbf{A} + \alpha\mathbf{I}$ is primitive, then $\mathbf{A} + \alpha\mathbf{I}$ must be nonnegative and irreducible, and hence $a_{ij} \geq 0$ for every $i \neq j$, and $\mathbf{A}$ must be irreducible (diagonal entries don't affect the reducibility or irreducibility).

(b) If $\mathbf{A}$ is essentially positive, then $\mathbf{A} + \alpha\mathbf{I}$ is primitive for some α (by the first part), so $(\mathbf{A} + \alpha\mathbf{I})^k > \mathbf{0}$ for some k. Consequently, for all $t > 0$,

$$\mathbf{0} < \sum_{k=0}^{\infty} \frac{t^k(\mathbf{A} + \alpha\mathbf{I})^k}{k!} = \mathrm{e}^{t(\mathbf{A}+\alpha\mathbf{I})} = \mathrm{e}^{t\mathbf{A}}\mathrm{e}^{t\alpha\mathbf{I}} = \mathbf{B} \implies \mathbf{0} < \mathrm{e}^{-\alpha t}\mathbf{B} = \mathrm{e}^{t\mathbf{A}}.$$

Conversely, if $\mathbf{0} < \mathrm{e}^{t\mathbf{A}} = \sum_{k=0}^{\infty} t^k\mathbf{A}^k/k!$ for all $t > 0$, then $a_{ij} \geq 0$ for every $i \neq j$, for if $a_{ij} < 0$ for some $i \neq j$, then there exists a sufficiently small $t > 0$ such that $[\mathbf{I} + t\mathbf{A} + t^2\mathbf{A}^2/2 + \cdots]_{ij} < 0$, which is impossible. Furthermore, $\mathbf{A}$ must be irreducible; otherwise

$$\mathbf{A} \sim \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{0} & \mathbf{Z} \end{pmatrix} \implies \mathrm{e}^{t\mathbf{A}} = \sum_{k=0}^{\infty} t^k\mathbf{A}^k/k! \sim \begin{pmatrix} \star & \star \\ \mathbf{0} & \star \end{pmatrix}, \quad \text{which is impossible.}$$

8.3.13. (a) Being essentially positive implies that there exists some $\alpha \in \Re$ such that $\mathbf{A}+\alpha\mathbf{I}$ is nonnegative and irreducible (by Exercise 8.3.12). If $(r, \mathbf{x})$ is the Perron eigenpair for $\mathbf{A} + \alpha\mathbf{I}$, then for $\xi = r - \alpha$, $(\xi, \mathbf{x})$ is an eigenpair for $\mathbf{A}$.

(b) Every eigenvalue of $\mathbf{A} + \alpha\mathbf{I}$ has the form $z = \lambda + \alpha$, where $\lambda \in \sigma(\mathbf{A})$, so if r is the Perron root of $\mathbf{A} + \alpha\mathbf{I}$, then for $z \neq r$,

$$|z| < r \implies \operatorname{Re}(z) < r \implies \operatorname{Re}(\lambda + \alpha) < r \implies \operatorname{Re}(\lambda) < r - \alpha = \xi.$$

(c) If $\mathbf{A} \leq \mathbf{B}$, then $\mathbf{A} + \alpha\mathbf{I} \leq \mathbf{B} + \alpha\mathbf{I}$, so Wielandt's theorem (p. 675) insures that $r_A = \rho(\mathbf{A} + \alpha\mathbf{I}) \leq \rho(\mathbf{B} + \alpha\mathbf{I}) = r_B$, and hence $\xi_A = r_A - \alpha \leq r_B - \alpha = \xi_B$.

8.3.14. If $\mathbf{A}$ is primitive with $r = \rho(\mathbf{A})$, then, by (8.3.10) on p. 674,

$$\begin{aligned}
\left(\frac{\mathbf{A}}{r}\right)^k \to \mathbf{G} > \mathbf{0} &\implies \exists\, k_0 \text{ such that } \left(\frac{\mathbf{A}}{r}\right)^m > \mathbf{0} \quad \forall m \geq k_0 \\
&\implies \frac{a_{ij}^{(m)}}{r^m} > 0 \quad \forall m \geq k_0 \\
&\implies \lim_{m\to\infty} \left(\frac{a_{ij}^{(m)}}{r^m}\right)^{1/m} \to 1 \implies \lim_{m\to\infty} \left[a_{ij}^{(m)}\right]^{1/m} = r.
\end{aligned}$$

Conversely, we know from the Perron–Frobenius theorem that $r > 0$, so if $\lim_{k\to\infty}\left[a_{ij}^{(k)}\right]^{1/k} = r$, then $\exists\ k_0$ such that $\forall m \geq k_0$, $\left[a_{ij}^{(m)}\right]^{1/m} > 0$, which implies that $\mathbf{A}^m > \mathbf{0}$, and thus $\mathbf{A}$ is primitive by Frobenius's test (p. 678).

Solutions for exercises in section 8. 4

8.4.1. The left-hand Perron vector for $\mathbf{P}$ is $\boldsymbol{\pi}^T = (10/59, 4/59, 18/59, 27/59)$. It's the limiting distribution in the regular sense because $\mathbf{P}$ is primitive (it has a positive diagonal entry—recall Example 8.3.3 (p. 678)).

8.4.2. The left-hand Perron vector is $\boldsymbol{\pi}^T = (1/n)(1, 1, \ldots, 1)$. Thus the limiting distribution is the uniform distribution, and in the long run, each state is occupied an equal proportion of the time. The limiting matrix is $\mathbf{G} = (1/n)\mathbf{e}\mathbf{e}^T$.

8.4.3. If $\mathbf{P}$ is irreducible, then $\rho(\mathbf{P}) = 1$ is a simple eigenvalue for $\mathbf{P}$, so

$$rank\,(\mathbf{I}-\mathbf{P}) = n - \dim N(\mathbf{I}-\mathbf{P}) = n - geo\ mult_{\mathbf{P}}(1) = n - alg\ mult_{\mathbf{P}}(1) = n-1.$$

8.4.4. Let $\mathbf{A} = \mathbf{I}-\mathbf{P}$, and recall that $rank\,(\mathbf{A}) = n-1$ (Exercise 8.4.3). Consequently,

$$\mathbf{A}\text{ singular} \Longrightarrow \mathbf{A}[\text{adj}\,(\mathbf{A})] = \mathbf{0} = [\text{adj}\,(\mathbf{A})]\mathbf{A} \quad \text{(Exercise 6.2.8, p. 484)},$$

and

$$rank\,(\mathbf{A}) = n-1 \Longrightarrow rank\,(\text{adj}\,(\mathbf{A})) = 1 \quad \text{(Exercises 6.2.11)}.$$

It follows from $\mathbf{A}[\text{adj}\,(\mathbf{A})] = \mathbf{0}$ and the Perron–Frobenius theorem that each column of $[\text{adj}\,(\mathbf{A})]$ must be a multiple of $\mathbf{e}$ (the column of 1's or, equivalently, the right-hand Perron vector for $\mathbf{P}$), so $[\text{adj}\,(\mathbf{A})] = \mathbf{e}\mathbf{v}^T$ for some vector $\mathbf{v}$. But $[\text{adj}\,(\mathbf{A})]_{ii} = P_i$ forces $\mathbf{v}^T = (P_1, P_2, \ldots, P_n)$. Similarly, $[\text{adj}\,(\mathbf{A})]\mathbf{A} = \mathbf{0}$ insures that each row in $[\text{adj}\,(\mathbf{A})]$ is a multiple of $\boldsymbol{\pi}^T$ (the left-hand Perron vector of $\mathbf{P}$), and hence $\mathbf{v}^T = \alpha\boldsymbol{\pi}^T$ for some α. This scalar α can't be zero; otherwise $[\text{adj}\,(\mathbf{A})] = \mathbf{0}$, which is impossible because $rank\,(\text{adj}\,(\mathbf{A})) = 1$. Therefore, $\mathbf{v}^T\mathbf{e} = \alpha \neq 0$, and $\mathbf{v}^T/(\mathbf{v}^T\mathbf{e}) = \mathbf{v}^T/\alpha = \boldsymbol{\pi}^T$.

8.4.5. If $\mathbf{Q}_{k\times k}$ $(1 \leq k < n)$ is a principal submatrix of $\mathbf{P}$, then there is a permutation matrix $\mathbf{H}$ such that $\mathbf{H}^T\mathbf{P}\mathbf{H} = \begin{pmatrix}\mathbf{Q} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{pmatrix} = \widetilde{\mathbf{P}}$. If $\mathbf{B} = \begin{pmatrix}\mathbf{Q} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$, then $\mathbf{B} \leq \widetilde{\mathbf{P}}$, and we know from Wielandt's theorem (p. 675) that $\rho(\mathbf{B}) \leq \rho\left(\widetilde{\mathbf{P}}\right) = 1$, and if $\rho(\mathbf{B}) = \rho\left(\widetilde{\mathbf{P}}\right) = 1$, then there is a number ϕ and a nonsingular diagonal matrix $\mathbf{D}$ such that $\mathbf{B} = e^{i\phi}\mathbf{D}\widetilde{\mathbf{P}}\mathbf{D}^{-1}$ or, equivalently, $\widetilde{\mathbf{P}} = e^{-i\phi}\mathbf{D}\mathbf{B}\mathbf{D}^{-1}$. But this implies that $\mathbf{X} = \mathbf{0}$, $\mathbf{Y} = \mathbf{0}$, and $\mathbf{Z} = \mathbf{0}$, which is impossible because $\mathbf{P}$ is irreducible. Therefore, $\rho(\mathbf{B}) < 1$, and thus $\rho(\mathbf{Q}) < 1$.

8.4.6. In order for $\mathbf{I}-\mathbf{Q}$ to be an M-matrix, it must be the case that $[\mathbf{I}-\mathbf{Q}]_{ij} \leq 0$ for $i \neq j$, and $\mathbf{I}-\mathbf{Q}$ must be nonsingular with $(\mathbf{I}-\mathbf{Q})^{-1} \geq \mathbf{0}$. It's clear that $[\mathbf{I}-\mathbf{Q}]_{ij} \leq 0$ because $0 \leq q_{ij} \leq 1$. Exercise 8.4.5 says that $\rho(\mathbf{Q}) < 1$, so the Neumann series expansion (p. 618) insures that $\mathbf{I}-\mathbf{Q}$ is nonsingular and $(\mathbf{I}-\mathbf{Q})^{-1}=\sum_{j=1}^{\infty}\mathbf{Q}^j \geq \mathbf{0}$. Thus $\mathbf{I}-\mathbf{Q}$ is an M-matrix.

8.4.7. We know from Exercise 8.4.6 that every principal submatrix of order $1 \leq k < n$ is an M-matrix, and M-matrices have positive determinants by (7.10.28) on p. 626.

8.4.8. You can consider an absorbing chain with eight states

$$\{(1,1,1),(1,1,0),(1,0,1),(0,1,1),(1,0,0),(0,1,0),(0,0,1),(0,0,0)\}$$

similar to what was described in Example 8.4.5, or you can use a four-state chain in which the states are defined to be the *number* of controls that hold at each activation of the system. Using the eight-state chain yields the following mean-time-to-failure vector.

$$\begin{matrix}(1,1,1)\\(1,1,0)\\(1,0,1)\\(0,1,1)\\(1,0,0)\\(0,1,0)\\(0,0,1)\end{matrix}\begin{pmatrix}368.4\\366.6\\366.6\\366.6\\361.3\\361.3\\361.3\end{pmatrix}=(\mathbf{I}-\mathbf{T}_{11})^{-1}\mathbf{e}.$$

8.4.9. This is a Markov chain with nine states (c,m) in which c is the chamber occupied by the cat, and m is the chamber occupied by the mouse. There are three absorbing states—namely $(1,1)$, $(2,2)$, $(3,3)$. The transition matrix is

$$\mathbf{P}=\frac{1}{72}\begin{matrix}(1,2)\\(1,3)\\(2,1)\\(2,3)\\(3,1)\\(3,2)\\(1,1)\\(2,2)\\(3,3)\end{matrix}\overset{\begin{matrix}(1,2) & (1,3) & (2,1) & (2,3) & (3,1) & (3,2) & (1,1) & (2,2) & (3,3)\end{matrix}}{\begin{pmatrix}18 & 12 & 3 & 6 & 3 & 9 & 6 & 9 & 6\\12 & 18 & 3 & 9 & 3 & 6 & 6 & 6 & 9\\3 & 3 & 18 & 9 & 12 & 6 & 6 & 9 & 6\\4 & 6 & 6 & 18 & 4 & 8 & 2 & 12 & 12\\3 & 3 & 12 & 6 & 18 & 9 & 6 & 6 & 9\\6 & 4 & 4 & 8 & 6 & 18 & 2 & 12 & 12\\0 & 0 & 0 & 0 & 0 & 0 & 72 & 0 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 72 & 0\\0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 72\end{pmatrix}}$$

The expected number of steps until absorption and absorption probabilities are

$$(\mathbf{I}-\mathbf{T}_{11})^{-1}\mathbf{e}=\begin{matrix}(1,2)\\(1,3)\\(2,1)\\(2,3)\\(3,1)\\(3,2)\end{matrix}\begin{pmatrix}3.24\\3.24\\3.24\\2.97\\3.24\\2.97\end{pmatrix} \quad \text{and} \quad (\mathbf{I}-\mathbf{T}_{11})^{-1}\mathbf{T}_{12}=\overset{\begin{matrix}(1,1) & (2,2) & (3,3)\end{matrix}}{\begin{pmatrix}0.226 & 0.41 & 0.364\\0.226 & 0.364 & 0.41\\0.226 & 0.41 & 0.364\\0.142 & 0.429 & 0.429\\0.226 & 0.364 & 0.41\\0.142 & 0.429 & 0.429\end{pmatrix}}$$