that the trace is the sum of the eigenvalues (recall (7.1.7)) to conclude that $\mathit{trace}\,(\mathbf{T}) = \sum_{i,j} \lambda_i \mu_j = \sum_i \lambda_i \sum_j \mu_j = \mathit{trace}\,(\mathbf{P})\,\mathit{trace}\,(\mathbf{Q})$.

7.1.22. (a) Use (6.2.3) to compute the characteristic polynomial for $\mathbf{D} + \alpha\mathbf{v}\mathbf{v}^T$ to be

$$\begin{aligned} p(\lambda) &= \det\left(\mathbf{D} + \alpha\mathbf{v}\mathbf{v}^T - \lambda\mathbf{I}\right) \\ &= \det\left(\mathbf{D} - \lambda\mathbf{I} + \alpha\mathbf{v}\mathbf{v}^T\right) \\ &= \det\left(\mathbf{D} - \lambda\mathbf{I}\right)\left(1 + \alpha\mathbf{v}^T(\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{v}\right) \qquad (\ddagger) \\ &= \left(\prod_{j=1}^{n}(\lambda - \lambda_j)\right)\left(1 + \alpha\sum_{i=1}^{n}\frac{v_i^2}{\lambda_i - \lambda}\right) \\ &= \prod_{j=1}^{n}(\lambda - \lambda_j) + \alpha\sum_{i=1}^{n}\left(v_i\prod_{j\neq i}(\lambda - \lambda_j)\right). \end{aligned}$$

For each λ_k, it is true that

$$p(\lambda_k) = \alpha v_k \prod_{j\neq k}(\lambda_k - \lambda_j) \neq 0,$$

and hence no λ_k can be an eigenvalue for $\mathbf{D} + \alpha\mathbf{v}\mathbf{v}^T$. Consequently, if ξ is an eigenvalue for $\mathbf{D} + \alpha\mathbf{v}\mathbf{v}^T$, then $\det(\mathbf{D} - \xi\mathbf{I}) \neq 0$, so $p(\xi) = 0$ and $(\ddagger)$ imply that

$$0 = 1 + \alpha\mathbf{v}^T(\mathbf{D} - \xi\mathbf{I})^{-1}\mathbf{v} = 1 + \alpha\sum_{i=1}^{n}\frac{v_i^2}{\lambda_i - \xi} = f(\xi).$$

(b) Use the fact that $f(\xi_i) = 1 + \alpha\mathbf{v}^T(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v} = 0$ to write

$$\begin{aligned} \left(\mathbf{D} + \alpha\mathbf{v}\mathbf{v}^T\right)(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v} &= \mathbf{D}(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v} + \mathbf{v}\left(\alpha\mathbf{v}^T(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v}\right) \\ &= \mathbf{D}(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v} - \mathbf{v} \\ &= \left(\mathbf{D} - (\mathbf{D} - \xi_i\mathbf{I})\right)(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v} \\ &= \xi_i(\mathbf{D} - \xi_i\mathbf{I})^{-1}\mathbf{v}. \end{aligned}$$

7.1.23. (a) If $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$, then

$$\ln p(\lambda) = \sum_{i=1}^{n}\ln(\lambda - \lambda_i) \implies \frac{p'(\lambda)}{p(\lambda)} = \sum_{i=1}^{n}\frac{1}{(\lambda - \lambda_i)}.$$

(b) If $|\lambda_i/\lambda| < 1$, then we can write

$$(\lambda - \lambda_i)^{-1} = \left(\lambda\left(1 - \frac{\lambda_i}{\lambda}\right)\right)^{-1} = \frac{1}{\lambda}\left(1 - \frac{\lambda_i}{\lambda}\right)^{-1} = \frac{1}{\lambda}\left(1 + \frac{\lambda_i}{\lambda} + \frac{\lambda_i^2}{\lambda^2} + \cdots\right).$$