(b) $\lambda = (3 \pm \sqrt{3})/2$, and the normal modes are determined by the corresponding eigenvectors, which are found in the usual way by solving

$$(\mathbf{K} - \lambda \mathbf{M})\mathbf{v} = \mathbf{0}.$$

They are

$$\mathbf{v}_1 = \begin{pmatrix} -1-\sqrt{3} \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1+\sqrt{3} \\ 1 \end{pmatrix}$$

(c) This part is identical to that in Example 7.6.1 (p. 559) except a 2×2 matrix is used in place of a 3×3 matrix.

7.6.3. Each mass "feels" only the spring above and below it, so

$$\begin{aligned} m_1 y_1'' &= \text{Force up} - \text{Force down} = ky_1 - k(y_2 - y_1) = k(2y_1 - y_2) \\ m_2 y_2'' &= \text{Force up} - \text{Force down} = k(y_2 - y_1) - k(y_3 - y_2) = k(-y_1 + 2y_2 - y_3) \\ m_3 y_3'' &= \text{Force up} - \text{Force down} = k(y_3 - y_2) \end{aligned}$$

(b) Gerschgorin's theorem (p. 498) shows that the eigenvalues are nonnegative, as since $\det(\mathbf{K}) \neq 0$, it follows that $\mathbf{K}$ is positive definite.

(c) The same technique used in the vibrating beads problem in Example 7.6.1 (p. 559) shows that modes are determined by the eigenvectors. Some computation is required to produce $\lambda_1 \approx .198$, $\lambda_2 \approx 1.55$, and $\lambda_3 \approx 3.25$. The modes are defined by the associated eigenvectors

$$\mathbf{x}_1 = \begin{pmatrix} \gamma \\ \alpha \\ \beta \end{pmatrix} \approx \begin{pmatrix} .328 \\ .591 \\ .737 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} -\beta \\ -\gamma \\ \alpha \end{pmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{pmatrix} -\alpha \\ \beta \\ -\gamma \end{pmatrix}.$$

7.6.4. Write the quadratic form as $13x^2 + 10xy + 13y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{z}^T \mathbf{A} \mathbf{z}$. We know from Example 7.6.3 on p. 567 that if $\mathbf{Q}$ is an orthogonal matrix such that $\mathbf{Q}^T \mathbf{A} \mathbf{Q} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, and if $\mathbf{w} = \mathbf{Q}^T \mathbf{z} = \begin{pmatrix} u \\ v \end{pmatrix}$, then

$$13x^2 + 10xy + 13y^2 = \mathbf{z}^T \mathbf{A} \mathbf{z} = \mathbf{w}^T \mathbf{D} \mathbf{w} = \lambda_1 u^2 + \lambda_2 v^2.$$

Computation reveals that $\lambda_1 = 8$, $\lambda_2 = 18$, and $\mathbf{Q} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, so the graph of $13x^2 + 10xy + 13y^2 = 72$ is the same as that for $18u^2 + 8v^2 = 72$ or, equivalently, $u^2/9 + v^2/4 = 1$. It follows from (5.6.13) on p. 326 that the uv-coordinate system results from rotating the standard xy-coordinate system counterclockwise by 45°.

7.6.5. Since $\mathbf{A}$ is symmetric, the LDU factorization is really $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$ (see Exercise 3.10.9 on p. 157). In other words, $\mathbf{A} \cong \mathbf{D}$, so Sylvester's law of inertia guarantees that the inertia of $\mathbf{A}$ is the same as the inertia of $\mathbf{D}$.